Republic of Iraq
Ministry of Education
General Directorate of Curricula

MATHEMATICS

6

Scientific Secondary Sixth

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Thirteenth Translated Edition **2025/1447**

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Introduction

The maths subject is considered one of the basic courses that helps students to acquaive educational abilities to develop thier thinking and solving problems and it helping to deal with difficult situations in thier life .

As a starting point of attention by the Ministry of Education represented by the General Directorate of Curricula to develop the curricula in general and specially of maths in order to go along with the technological scientific .

The book consists of six chapters:

The first chapter is (Complex Numbers) which contains:

Find the roots and their properties ,Solving second –order equation .in complex number , polar coordinates , Modulus and Argument of complex numbers .

The second chapter is (Conic Sections) which contains:

The standard equation .for Parabola , Ellipse and Hyperbola, and eccentricity of each them.

The third chapter is (Application of Differentiation) which contains: Related Rates.

Rolle's and Mean value theorem ,the derivative test for increasing and decreasing for a function, local Max and Min, Concavity and inflection opint, the second derivative test for local Max. and Min graphing functions and optimization (Max ,Min) problems.

The fourth chapter is (Integration)which contains: Integration and its applications, find integration of Algebraic, Logarithmic , Exponential and Circular functions find the area between x-axis and the curve , and the area between two curves ,find the volume of revolution .

The fifth chapter is (Ordinary Differential Equation) with contains: find (Degree, Order and Solution) and solving equation by separation of variable.

The sixth chapter is (Space Geometry) which contains: Dihderd angle and perpendicular planes ,and Orthogonal Projection on plane some.

We hope God help us to serve our country and our sons

Authors

01 COMPLEX NUMBERS

02 CONIC SECTION

03 APPLICATION OF DIFFERENTIATIONS

04 INTEGRATION

05 ORDINATY DIFFERENTIAL EQUATONS

06 SPACE GEOMETRY

01 COMPLEX NUMBERS

Chapter 1 : Complex Numbers

1 – 1 Need to expand set of real Numbers.

1 – 2 Operations on complex Numbers set.

1 – 3 Complex Conjugate.

1 – 4 Square roots of complex Number.

1 – 5 Solution of quadratic equation in C.

1 – 6 Cubic roots for one integer.

1 – 7 Geometrical representation for complex Numbers.

1 – 8 Polar form of complex Number.

1 – 9 De Moivre's Theorem.

Terminology

Term	Symbol or Mathematical Relation
Real part of number Z : R(Z)	$R(Z) = x = r . \cos\theta$
Imaginary part of number Z : I(Z)	$I(Z) = y = r . \sin\theta$
Argument of complex number Z	$\arg(Z) = \theta$
Modulus of complex number Z	$r = \|Z\| = \text{mod } Z$
Left hand side	LHS
Right hand side	RHS
Naturel numbers	N
Integers	Z
Whole numbers	w
Rational numbers	Q
Real numbers	R
Complex numbers	C

1 – 1 : Need to Expand set of real numbers

We have previously studied solution of linear equation. There is one unique solution for a set of real numbers for any linear equation.

When studying quadratic equations, some of them have a solution in the real number set, some do not have solution in this set, like equations :

$(x^2 + 1 = 0)$, $(x^2 + 4x + 5 = 0)$

As you learned, quadratic equations whose discriminant is $(b^2 - 4ac)$ a negative number., have no solution in the real numbers set.

Emergence of this type of equations in physical and geometrical applications has urged the need to expand real numbers set to a larger set, which is the complex number that need to be the subject of our discussion in this chapter.

When we want to solve the equation $(x^2 + 1 = 0)$ or $(x^2 = -1)$, there is no real number whose sqaure is (–1), so we suppose a number $\sqrt{-1}$. It is a non-real number (i), it is called imaginary unit, it is not a number that can be associated with computing or measurement.

(i) Satisfies algebraic properties of the real numbers except for order property (ordinary), therefore we can compute the powers of (i) as follows :

$i^2 = -1$

$i^3 = i^2 . i = (-1) . i = -i$

$i^4 = i^2 . i^2 = (-1) . (-1) = 1$

$i^{27} = i^{26} . i = (i^2)^{13} . i = (-1)^{13} . i = -i$

$i^{81} = i^{80} . i = (i^2)^{40} . i = (-1)^{40} . i = 1 . i = i$

$i^{-7} = (i)^{-8} . i = (i^2)^{-4} . i = (-1)^{-4} . i = i$

$i^{-15} = i^{-16} . i = (i^2)^{-8} . i = (-1)^{-8} . i = i$

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Generally :

$i^{4n+r} = i^r$, $n \in w$, $r = 0, 1, 2, 3$

$w = \{0, 1, 2, 3, \ldots\}$

This means when (i) is power to a positive integer, the result is one of set $\{-i, i, -1, 1\}$

Whereby (i) power is divided by 4, the remainder is new power to (i)

For example :

$i^{25} = i$ because 25 quotient by 4 is 6 and the remainder is 1.

$i^{99} = i^3 = -i$ because 99 quotient by 4 is 24 and the remainder is 3.

Example 1

Write the following in simplest form :

a) i^{16}

b) i^{58}

c) i^{12n+93}

d) i^{-13}

Solution

a) $i^{16} = i^{4(4)+0} = i^0 = 1$

b) $i^{58} = i^{4(14)+2} = i^2 = -1$

c) $i^{12n+93} = i^{4(3n)} \cdot i^{93} = (1)^{3n} \cdot i^{4(23)+1} = (1)(i) = i$

d) $i^{-13} = \frac{1}{i^{13}} = \frac{i^{16}}{i^{13}} = i^3 = -i$

Note

The square root of a real negative number can be written using (i) significance.

For example:

$\sqrt{-16} = \sqrt{16}.\sqrt{-1} = 4i$

$\sqrt{-25} = \sqrt{25}.\sqrt{-1} = 5i$

$\sqrt{-12} = \sqrt{12}.\sqrt{-1} = 2\sqrt{3}\,i$

$\sqrt{-15} = \sqrt{15}.\sqrt{-1} = \sqrt{15}\,i$

Generally :

$\sqrt{-a} = \sqrt{a}.\sqrt{-1} = \sqrt{a}\,i \;\; , \forall\, a \geq 0$

Now after laerning what the imaginary number is, What the number (a + bi) is called? Whereby (a) is real number, (b) is real number $\sqrt{-1} = i$?

Definition 1-1-1 **Complex Number**

The number $c = a + bi$, whereby a, b are real numbers, $\sqrt{-1} = i$ is a complex number, its real part is (a) while (b) its imaginary part. The complex number set is symbolized $\mathbb{C}$, the form $a + bi$ is called the standard form or the algebraic form of the complex number.

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Note

Any complex number $c = a + bi$ can be made equivalent for unique order pair (a, b). a, b a real numbers, conversely, the real number (a) can be written as $a + 0i$ or $(a, 0)$ and (i) is the imaginary unit, such that :

$\mathbf{i \Leftrightarrow (0, 1)}$ **or** $\mathbf{i = 0 + 1i}$

The number $(0, b) \Leftrightarrow bi$ is a pure imaginary number, as for the number

$\mathbf{(a, 0) \Leftrightarrow a = a + 0i}$, it is a pure real number.

The number $-2 + 3i$ is a complex number, its real part is –2, imaginary part is 3.

The number -2 is a complex number, its real part is –2, imaginary part is 0.

The number $-3i$ is a complex number, its real part is 0, imaginary part is –3.

Example 2

Write the following numbers in the form of $a + bi$:

a) -5 b) $\sqrt{-100}$ c) $-1-\sqrt{-3}$ d) $\dfrac{1+\sqrt{-25}}{4}$

Solution

a) $-5 = -5 + 0i$

b) $\sqrt{-100} = \sqrt{100}.\sqrt{-1} = 10i = 0 + 10i$

c) $-1-\sqrt{-3} = -1 - \sqrt{3}\, i$

d) $\dfrac{1+\sqrt{-25}}{4} = \dfrac{1}{4} + \dfrac{\sqrt{25}\, i}{4} = \dfrac{1}{4} + \dfrac{5}{4} i$

Since each real number can be written as $a + 0i$ or $(a, 0)$, i.e, it can be written as a complex number, its imaginary part is zero, this means:

Note

Real number set R is a subset of the complex numbers C, this means $R \subset C$.

Definition 1-1-2 Equality of Complex Numbers

If $c_1 = a_1 + b_1 i$, $c_2 = a_2 + b_2 i$ then $b_1 = b_2$, $a_1 = a_2 \Leftrightarrow c_1 = c_2$

The two complex numbers are equal if their real parts are equal and their imaginary parts are equal and vice versa

Example 3

Find the value of real x, y that satisfies the equation in the following :

a) $2x - 1 + 2i = 1 + (y + 1)i$

b) $3x + 4i = 2 + 8yi$

c) $(2y + 1) - (2x - 1)i = -8 + 3i$

Solution

a) $2x - 1 + 2i = 1 + (y + 1)i$

$2x - 1 = 1 \Rightarrow 2x = 2$

$\Rightarrow x = 1$

$2 = y + 1 \Rightarrow y = 2 - 1$

$y = 1$

b) $3x + 4i = 2 + 8yi$

$3x = 2 , 4 = 8y \Rightarrow x = \frac{2}{3} , y = \frac{4}{8} = \frac{1}{2}$

c) $(2y+1)-(2x-1)i=-8+3i$

$2y+1=-8\ ,\ -(2x-1)=3\ \Rightarrow$

$2y=-9\ ,\ -2x=2\ \Rightarrow$

$y=\frac{-9}{2}\ ,\ x=-1$

1 – 2 Operations on complex numbers set :

First: Addition operation on complex numbers Set:

Definition 1-2-1 Addition of Complex Numbers

Let $c_1=a_1+b_1i$, $c_2=a_2+b_2i$ **whereby** c_1 , $c_2 \in C$ **then,**

$c_1+c_2=(a_1+a_2)+(b_1+b_2)i$

As you know, $(a_1+a_2)\in R$, $(b_1+b_2)\in R$, because real number set is closed under addition operation.

$\therefore\ (a_1+a_2)+(b_1+b_2)i \in C$

Complex numbers set is closed under addition operation

Example 4

Add two complex numbers in the following :

a) $3+4\sqrt{2}i$, $5-2\sqrt{2}i$

b) 3 , $2-5i$

c) $1-i$, $3i$

Solution

a) $(3 + 4\sqrt{2}\,i) + (5 - 2\sqrt{2}\,i) = (3 + 5) + (4\sqrt{2} - 2\sqrt{2})i$
$= 8 + 2\sqrt{2}\,i$

b) $(3) + (2 - 5i) = (3 + 0i) + (2 - 5i)$
$= (3 + 2) + (0 - 5)i = 5 - 5i$

c) $(1 - i) + 3i = (1 - i) + (0 + 3i)$
$= (1 + 0) + (-1 + 3)i = 1 + 2i$

Properties of addition operation in complex numbers

The addition operation in complex numbers has the following properties :

$\forall\, c_1, c_2, c_3 \in C$ then,

- **Commutativity :** $c_1 + c_2 = c_2 + c_1$
- **Associativity :** $c_1 + (c_2 + c_3) = (c_1 + c_2) + c_3$
- **Additive Inverse :** $\forall\, c \in C,\ c = a + bi\ \ \exists\ z : c + z = z + c = 0 \Rightarrow z = -c$

Whereby **$-a - bi = -c$, $(-c)$ is called the additive inverse of the complex number c.**

- **Additive Identity : symbolized (e) and defined as :** $e = 0 = 0 + 0i \in C$

Note

Subtracting a complex number from the other equals the addition of the first complex number with the additive inverse of the second complex number.

Example 5

Find the result of

$(7 - 13i) - (9 + 4i)$

Solution

$$(7 - 13i) - (9 + 4i) = (7 - 13i) + (-9 - 4i)$$
$$= (7 - 9) + (-13 - 4)i$$
$$= -2 - 17i$$

Example 6

Solve the equation

$(2 - 4i) + x = -5 + i$ whereby $x \in C$

Solution

$(2 - 4i) + x = -5 + i$

By adding the addivite inverse of the number (2 – 4i) to the two sides :

$$(2 - 4i) + (-2 + 4i) + x = (-5 + i) + (-2 + 4i)$$
$$x = (-5 + i) + (-2 + 4i)$$
$$= (-5 - 2) + (1 + 4)i$$
$$= -7 + 5i$$

Second : Multiplication operation on complex numbers set

To multiply two complex numbers, they are multiplied as two algebraic values and substitude instead of i^2 by the number (–1), as follows :

If $c_1 = a_1 + b_1 i$, $c_2 = a_2 + b_2 i$ then

$$c_1 . c_2 = (a_1 + b_1 i) . (a_2 + b_2 i)$$
$$= a_1.a_2 + a_1.b_2 i + a_2.b_1 i + b_1.b_2 i^2$$
$$= a_1.a_2 + a_1.b_2 i + a_2.b_1 i - b_1.b_2$$
$$= (a_1.a_2 - b_1.b_2) + (a_1.b_2 + a_2.b_1)i$$

Note

If $k \in R$, $c=a+bi$ then $kc=ka+k.bi$

Definition 1-2-2 Multiplying Complex Numbers

Let $c_1=a_1+b_1 i$, $c_2=a_2+b_2 i$ whereby $c_1, c_2 \in C$ then

$c_1 . c_2 = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1) i$

As you know $(a_1 a_2 - b_1 b_2) \in R$ and $(a_1 b_2 + a_2 b_1) \in R$ because

R is closed under multiplication operation

Therefore; $c_1 . c_2 \in C$

Complex numbers set is closed under multiplication operation.

Example 7

Find the result of the following

a) $(2 - 3i)(3 - 5i)$

b) $(3 + 4i)^2$

c) $i(1 + i)$

d) $-\frac{5}{2}(4 + 3i)$

e) $(1 + i)^2 + (1 - i)^2$

Solution

a) $(2-3i)(3-5i) = (6-15) + (-10-9)i$

$= -9 - 19i$

Or the result can be reached by distribution :

$(2-3i)(3-5i) = 6 - 10i - 9i + 15i^2 = -9 - 19i$

b) $(3+4i)^2 = 9 + 24i + 16i^2$

$= 9 + 24i - 16$

$= -7 + 24i$

Or the result can be reached by distribution :

$(3+4i)^2 = (3+4i)(3+4i) = 9 + 12i + 12i + 16i^2$

$= (9-16) + (12+12)i$

$= -7 + 24i$

c) $i(1+i) = i + i^2 = -1 + i$

d) $-\frac{5}{2}(4+3i) = -10 - \frac{15}{2}i$

e) $(1+i)^2 + (1-i)^2 = (1 + 2i + i^2) + (1 - 2i + i^2)$

$= 2i + (-2i) = 0$

Properties of multiplication operation on complex numbers

$\forall\ c_1, c_2, c_3 \in C$

Multiplication operation of the complex numbers has the following properties :

- **Commutativity :** $c_1 \times c_2 = c_2 \times c_1$
- **Associativity :** $c_1 \times (c_2 \times c_3) = (c_1 \times c_2) \times c_3$
- **The Multiplicative Identity which is (1 +0i) = 1**

The Multiplicative Inverse :

$c \times \frac{1}{c} = (1 + 0i)$ whereby $\exists \frac{1}{c} \in C$, $\forall c \neq (0 + 0i)$

i.e. each complex number c other than zero has a multiplicative inverse

$\frac{1}{c}$ that belongs to complex numbers set.

1 – 3 Complex Conjugate

Definition 1-3-1 Complex Number Conjugate

Conjugate for complex number **c=a+bi** is **$\overline{c}$=a-bi** $\forall$ **a, b $\in$ R**

For example :

3+i is conjugate to number 3-i and vice versa in correct, also (i) conjugate is (–i) and the vice versa.

5-4i is conjugate to 5+4i and the vice versa, also 7 conjugate is 7.

Note

The conjugate, as appears from the definition, satisfies the following properties :

1) $\overline{c_1 \pm c_2} = \overline{c_1} \pm \overline{c_2}$

2) $\overline{c_1 \cdot c_2} = \overline{c_1} \cdot \overline{c_2}$

3) $\overline{\overline{c}} = c$

4) If $c = a + bi$ then $\overline{c} \cdot c = a^2 + b^2$

5) If $c \in R$ then $c = \overline{c}$

6) $\overline{\left(\frac{c_1}{c_2}\right)} = \frac{\overline{c_1}}{\overline{c_2}}$, $c_2 \neq 0$

Example 8

If $c_1=1+i$, $c_2=3-2i$, then , check :

1) $\overline{c_1 \pm c_2} = \overline{c_1} \pm \overline{c_2}$

2) $\overline{c_1 \cdot c_2} = \overline{c_1} \cdot \overline{c_2}$

Solution

1) $\overline{c_1 + c_2} = \overline{(1+i)+(3-2i)}$

$= \overline{(4-i)} = 4+i$

$\overline{c_1} + \overline{c_2} = \overline{(1+i)} + \overline{(3-2i)}$

$= (1-i)+(3+2i) = 4+i$

$\therefore \quad \overline{c_1 + c_2} = \overline{c_1} + \overline{c_2}$

Check yourself

$\overline{c_1 - c_2} = \overline{c_1} - \overline{c_2}$

2) $\overline{c_1 \cdot c_2} = \overline{(1+i)\cdot(3-2i)}$

$= \overline{3-2i+3i-2i^2} = \overline{5+i} = 5-i$

$\overline{c_1} \cdot \overline{c_2} = \overline{(1+i)}\,\overline{(3-2i)} = (1-i)(3+2i)$

$= (3+2)+(2-3)i = 5-i$

$\therefore \quad \overline{c_1 \cdot c_2} = \overline{c_1} \cdot \overline{c_2}$

Example 9

Find the multiplicative inverse of the number c=2-2i and put it in the standard form of the complex number.

Solution

The multiplicative inverse of number c is $\frac{1}{c}$

$$\frac{1}{c}=\frac{1}{2-2i}=\frac{1}{2-2i}\cdot\frac{2+2i}{2+2i}=\frac{2+2i}{4+4}=\frac{2+2i}{8}=\frac{1}{4}+\frac{1}{4}i$$

Example 10

If $\frac{3-2i}{i}$, $\frac{x-yi}{1+5i}$ are conjugated, then find value of each x, y ∈ R.

Solution

$$\frac{3-2i}{i}=\overline{\left(\frac{x-yi}{1+5i}\right)} \quad\Rightarrow\quad \frac{3-2i}{i}=\frac{x+yi}{1-5i}$$

$$xi+yi^2=3-15i-2i+10i^2$$

$$xi-y=-7-17i$$

$\therefore\ x=-17$ $\qquad -y=-7$

$y=7$

Example 11

If $c_1=3-2i$, $c_2=1+i$ the prove that :

$$\overline{\left(\frac{c_1}{c_2}\right)}=\frac{\overline{c_1}}{\overline{c_2}}$$

Solution

$$\overline{\left(\frac{c_1}{c_2}\right)}=\overline{\left(\frac{3-2i}{1+i}\right)}=\overline{\left(\frac{3-2i}{1+i}\times\frac{1-i}{1-i}\right)}$$

$$=\overline{\left(\frac{3-3i-2i+2i^2}{1+1}\right)}=\overline{\left(\frac{1-5i}{2}\right)}=\overline{\frac{1}{2}-\frac{5}{2}i}=\frac{1}{2}+\frac{5}{2}i$$

$$\frac{\overline{c_1}}{\overline{c_2}}=\frac{\overline{3-2i}}{\overline{1+i}}=\frac{3+2i}{1-i}$$

$$=\frac{3+2i}{1-i}\times\frac{1+i}{1+i}=\frac{3+3i+2i+2i^2}{1+1}$$

$$=\frac{1+5i}{2}=\frac{1}{2}+\frac{5}{2}i$$

$$\therefore\ \overline{\left(\frac{c_1}{c_2}\right)}=\frac{\overline{c_1}}{\overline{c_2}}$$

Note

To divide the complex number c_1 to the complex number c_2 where $c_2 \neq 0$, we multiply the numerator and denominator of the $\frac{c_1}{c_2}$ by the conjugate of denominator , then :

$$\frac{c_1}{c_2}=\frac{c_1}{c_2}\times\frac{\overline{c_2}}{\overline{c_2}}$$

Example 12

Write the following in the form of a+bi :

a) $\frac{1+i}{1-i}$ b) $\frac{2-i}{3+4i}$ c) $\frac{1+2i}{-2+i}$

Solution

a) $\frac{1+i}{1-i}=\frac{1+i}{1-i}\times\frac{1+i}{1+i}=\frac{1+2i+i^2}{1+1}=\frac{2i}{2}=i=0+i$

b) $\frac{2-i}{3+4i}=\frac{2-i}{3+4i}\times\frac{3-4i}{3-4i}=\frac{6-8i-3i+4i^2}{9+16}=\frac{2-11i}{25}=\frac{2}{25}-\frac{11}{25}i$

c) $\frac{1+2i}{-2+i}=\frac{1+2i}{-2+i}\times\frac{-2-i}{-2-i}=\frac{-2-i-4i-2i^2}{4+1}=\frac{-5i}{5}=-i=0-i$

Note

x^2+y^2 can be analyzed to product of two complex numbers, each has the form of a + bi :

$x^2+y^2=x^2-y^2i^2=(x-yi)(x+yi)$

Example 13

Analyze both numbers 10, 53 into product of two complex numbers in the form of a + bi where a, b are rational numbers.

$10 = 1 + 9$
$= 1 - 9i^2$
$= (1 - 3i)(1 + 3i)$ **or**

$10 = 9 + 1$
$= 9 - i^2$
$= (3 - i)(3 + i)$

$53 = 4 + 49$
$= 4 - 49i^2$
$= (2 - 7i)(2 + 7i)$ **or**

$53 = 49 + 4$
$= 49 - 4i^2$
$= (7 - 2i)(7 + 2i)$

01 COMPLEX NUMBERS

Exercises

1. Put the following complex numbers in standard form :

i^5 , i^6 , i^{124} , i^{999} , i^{4n+1} $\forall\, n \in w$, $(2 + 3i)^2 + (12 + 2i)$

$(10 + 3i)(0 + 6i)$, $(1 + i)^4 - (1 - i)^4$, $\frac{12+i}{i}$, $\frac{3+4i}{3-4i}$

$\frac{i}{2+3i}$, $\left(\frac{3+i}{1+i}\right)^3$, $\frac{2+3i}{1-i} \times \frac{1+4i}{4+i}$, $(1+i)^3+(1-i)^3$

2. Find the value of each x, y that satisfies the following equations :

a) $y + 5i = (2x + i)(x + 2i)$

b) $8i = (x + 2i)(y + 2i) + 1$

c) $\left(\frac{1-i}{1+i}\right) + (x + yi) = (1 + 2i)^2$

d) $\frac{2-i}{1+i}x + \frac{3-i}{2+i}y = \frac{1}{i}$

3. Prove the following :

a) $\frac{1}{(2-i)^2} - \frac{1}{(2+i)^2} = \frac{8}{25}i$

b) $\frac{(1-i)^2}{1+i} + \frac{(1+i)^2}{1-i} = -2$

c) $(1 - i)(1 - i^2)(1 - i^3) = 4$

4. Analyze each of numbers 29, 125, 41, 85 to product of two complex numbers in the form of a + bi where a, b are rational numbers.

5. Find value of real x, y if, $\frac{6}{x+yi}$, $\frac{3+i}{2-i}$ are conjugated.

1 – 4 Square roots of complex number

You have learned that if (a) is a real positive number, there are two real numbers, which are $\pm\sqrt{a}$, both satisfy the equation $x^2 = a$ and it is called $\pm\sqrt{a}$, the two square roots of the number (a). But if $a = 0$, it has only one root which is 0. Now we will study square roots of the complex number :

Example 14

Find the square roots of $c = 8 + 6i$

Solution

Let the square root of c be $x + yi$

$\therefore\ (x+yi)^2 = 8+6i$

$\Rightarrow\ x^2 + 2xyi + i^2y^2 = 8+6i$

$\Rightarrow\ (x^2 - y^2) + 2xyi = 8+6i$

$$\left.\begin{array}{l} x^2 - y^2 = 8 \text{.......................}(1) \\ 2xy = 6 \Rightarrow y = \dfrac{3}{x} \text{.............}(2) \end{array}\right\}$$ **The two complex numbers are equal .**

Then substitute equation 2 in 1 :

$$x^2 - \left(\frac{3}{x}\right)^2 = 8$$

$x^2 - \dfrac{9}{x^2} = 8$ **Multiplying product of two sides by $x^2 \neq 0$:**

$\Rightarrow\ x^4 - 8x^2 - 9 = 0$

$\Rightarrow\ (x^2 - 9)(x^2 + 1) = 0$

$\Rightarrow\ x = \pm 3$ or $(x^2 = -1)$ is neglected because $x \in R$.

Then we substitute in equation 2 the value of x , we have :

$$y = \frac{3}{\pm 3}$$

$$y = \pm 1$$

$$c_1 = 3 + i \ , \ c_2 = -3 - i$$

x	3	–3
y	1	–1

i.e. roots of number (c) are 3 + i, –3 – i

Example 15

Find square roots for : –25, –17, –i, 8i

Solution

a) Let $c^2 = -25$

$$c = \pm\sqrt{-25} = \pm\sqrt{25}\,i = \pm 5i$$

b) Let $c^2 = -17$

$$c = \pm\sqrt{-17} = \pm\sqrt{17}\,i$$

c) Let **(x + yi) is the square root for –i**

$\therefore\ (x + yi)^2 = -i \ \Rightarrow\ x^2 + 2xyi + y^2i^2 = 0 - i$

$x^2 - y^2 = 0$ .... (1)

$2xy = -1$

$\therefore\ y = \dfrac{-1}{2x}$ .... (2)

Then we substitute from equation (2) by equation (1) we have :

$$x^2 - \frac{1}{4x^2} = 0 \ \Rightarrow$$

By multiplying both sides by, $4x^2 \neq 0$ we have :

$4x^4 - 1 = 0$

$(2x^2 - 1)\ (2x^2 + 1) = 0$

Either $x^2 = -\frac{1}{2}$ (it is neglected because $x \in R$)

Or $x = \pm\frac{1}{\sqrt{2}}$ by substituting x value in equation (2), we have :

$\therefore\ y = -\left(\frac{1}{\pm(2)\frac{1}{\sqrt{2}}}\right) = \mp\frac{1}{\sqrt{2}}$

x	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$
y	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$

$\therefore$ **Square roots of $-i$ are :** $\pm\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$

d) Let $x + yi$ is the square root for $8i$:

$\therefore\ (x + yi)^2 = 8i \Rightarrow x^2 + 2xyi - y^2 = 0 + 8i$

$x^2 - y^2 = 0$ .... (1)

$2xy = 8$

$\therefore\ y = \frac{4}{x}$ .... (2)

By substitute equation (2) in equation (1) we have :

$x^2 - \frac{16}{x^2} = 0$

Multiplying both sides by, $x^2 \neq 0$ produces :

$x^4 - 16 = 0 \Rightarrow (x^2 - 4)(x^2 + 4) = 0 \Rightarrow$

Either $x^2 = -4$ (neglected because $x \in R$)

Or $x^2 = 4 \Rightarrow x = \pm 2$

By substitute x value in equation (2) we have :

$$y = \frac{4}{\pm 2} = \pm 2$$

x	2	-2
y	2	-2

∴ **Square roots of 8i are ± (2 + 2i)**

1 – 5 Solution of Quadratic Equation in C

You have learned in intermediate stage that for the equation $ax^2 + bx + c = 0$ (whereby $a \neq 0$ and a, b, c ∈ R) have two solutions in the Quadratic formula :

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

You know that if the discriminant value $b^2 - 4ac$ is negative, there is no real solution for the equation.But , there are two solutions in the complex numbers set.

Example 16

Solve the equation $x^2 + 4x + 5 = 0$ **in the set of complex numbers**

Solution

According to the quadratic formula :

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{16 - (4)(1)(5)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{16 - 20}}{2} = \frac{-4 \pm \sqrt{-4}}{2}$$

$$= \frac{-4 \pm 2i}{2} = -2 \pm i$$

i.e. the equation has two roots $-2 + i , -2 - i$

The solution set is { -2 + i , -2-i }

Note

Using the quadratic formula , we see that both roots of the quadratic equation $ax^2 + bx + c = 0$ whose coefficicents are real are :

$$x_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad , \quad x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

Addition of roots is : $x_1 + x_2 = \frac{-b}{a}$

Product of roots is : $x_1 \cdot x_2 = \frac{c}{a}$

These properties can be useful, as follows :

First : If $x + yi$, $y \neq 0$ was one of the roots of the equation, $ax^2 + bx + c = 0$, $a \neq 0$ $a, b, c \in R$, then, $x - yi$ is the other root.

Second : Division of both sides of equation, $ax^2 + bx + c = 0$ by $a \neq 0$, we have :

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$ which is :

x^2 – (the sum of two roots) x + product of two roots = 0

Example 17

Find the quadratic equation whose roots are ± (2 + 2i).

Solution

Sum of two roots : $(2 + 2i) + (-2 - 2i) = (2 - 2) + (2 - 2)i = 0$

Product of two roots : $(2 + 2i)(-2 - 2i) = -(2 + 2i)^2$

$= -(4 + 8i + 4i^2)$

$= -8i$

∴ **The quadratic equation is :** $x^2 - 0x + (-8i) = 0$

$\Rightarrow x^2 - 8i = 0$

$\Rightarrow x^2 = 8i$

Example 18

Find the quadratic equation whose coefficients are real and one of its roots is 3 – 4i

Solution

The coefficients are real and one of the root is 3 – 4i

The other conjugate root is 3 + 4i

Sum of two roots = 6

Product of two roots = 25

The equation is : $x^2 - 6x + 25 = 0$

Exercises

1. Solve the following quadratic equations and show which has conjugate roots?

a) $z^2 = -12$

b) $z^2 - 3z + 3 + i = 0$

c) $2z^2 - 5z + 13 = 0$

d) $z^2 + 2z + i(2 - i) = 0$

e) $4z^2 + 25 = 0$

f) $z^2 - 2zi + 3 = 0$

2. Find the quadratic equation whose roots are m, L. where :

a) $m = 1 + 2i$, $L = 1 - i$

b) $m = \frac{3-i}{1+i}$, $L = (3 - 2i)^2$

3. Evaluate the square roots for the following complex numbers :

a) $-6i$

b) $7 + 24i$

c) $\frac{4}{1-\sqrt{3}i}$

4. What is quadratic equation of a real coefficients and one of its roots is :

a) i

b) $5 - i$

c) $\frac{\sqrt{2}+3i}{4}$

5. If $3 + i$ was one root for equation $x^2 - ax + (5 + 5i) = 0$, then what is the value of $a \in C$? and what is the other root?

01 COMPLEX NUMBERS

(1 – 6) Cubic Roots of one Integer

Let $z^3 = 1$, then : $z^3 - 1 = 0 \Rightarrow (z-1)(z^2+z+1) = 0$

Either $\Rightarrow z - 1 = 0 \Rightarrow z = 1$

Or $\Rightarrow z^2 + z + 1 = 0$

To solve the equation $z^2 + z + 1 = 0$ we use the (Quadratic formula) :

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1 - 4(1)(1)}}{2(1)}$$

$$= \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1}{2} \pm \frac{\sqrt{3}}{2} i$$

i.e. the cubic roots for positive integer 1 are :

$$1 , -\frac{1}{2} + \frac{\sqrt{3}}{2} i , -\frac{1}{2} - \frac{\sqrt{3}}{2} i$$

Square of any of the imaginary roots equals the other imaginary root when conjugated (check it!).

If one imaginary root is ω (Omega) , then the other root is ω^2 Therefore, the cubic roots of integer 1 is written as :

$1, \omega , \omega^2$

These roots satisfy the following properties :

1) $1 + \omega + \omega^2 = 0$

2) $\omega^3 = 1$

From property 1, we get the following :

1) $\omega + \omega^2 = -1$

2) $1 + \omega = -\omega^2$

3) $1 + \omega^2 = -\omega$

4) $\omega = -1 - \omega^2$

5) $\omega^2 = -1 - \omega$

6) $1 = -\omega - \omega^2$

From property 2, we get the following :

$\omega^4 = \omega^3 . \omega = 1 . \omega = \omega$

$$\omega^{-4} = \frac{1}{\omega^4} = \frac{1}{\omega^3 \cdot \omega} = \frac{1}{\omega} = \frac{\omega^3}{\omega} = \omega^2$$

$\omega^5 = \omega^3 \cdot \omega^2 = 1 \cdot \omega^2 = \omega^2$

$$\omega^{-5} = \frac{1}{\omega^5} = \frac{1}{\omega^3 \cdot \omega^2} = \frac{1}{1 \cdot \omega^2} = \frac{\omega^3}{\omega^2} = \omega$$

$\omega^6 = (\omega^3)^2 = (1)^2 = 1$

$$\omega^{-6} = \frac{1}{\omega^6} = \frac{1}{1} = 1$$

Keeping in this pattern, (ω) powers for integer are one of these :

$1, \omega, \omega^2$

These values are repeated the as the exponents increase by (3) consecutively :

$\omega^{3n+r} = \omega^r$. Where n is an integer, r = 0, 1, 2

Example 19

Evaluate : ω^{33} , ω^{25} , ω^{-58}

$\omega^{33} = \omega^{3(11)+0} = \omega^0 = 1$

$\omega^{25} = \omega^{3(8)+1} = \omega^1 = \omega$

$\omega^{-58} = \omega^{3(-20)+2} = 1 \cdot \omega^2 = \omega^2$

i.e. Division remainder of (ω) exponent by (3) is the new exponent of ω

Example 20

Prove that :

a) $\omega^7 + \omega^5 + 1 = 0$

b) $(5 + 3\omega + 3\omega^2)^2 = -4(2 + \omega + 2\omega^2)^3 = 4$

Solution

a) $\text{LHS} = \omega^7 + \omega^5 + 1 = \omega^6 . \omega + \omega^3 . \omega^2 + 1$

$= \omega + \omega^2 + 1 = 0 = \text{RHS}$

(According to property 1)

b) the fist amount = $(5 + 3\omega + 3\omega^2)^2 = [5 + 3(\omega + \omega^2)]^2$

$= [5 - 3]^2 = (2)^2 = 4$

Also

the second amount = $-4(2 + \omega + 2\omega^2)^3 = -4[2(1 + \omega^2) + \omega]^3$

$= -4[-2\omega + \omega]^3 = -4[-\omega]^3$

$= -4(-1) = 4$

$\therefore\ (5 + 3\omega + 3\omega^2)^2 = -4(2 + \omega + 2\omega^2)^3 = 4$

Example 21

Find the quadratic equation whose roots are :

a) $1 - i\omega^2\ ,\ 1 - i\omega$

b) $\frac{2}{1-\omega}\ ,\ \frac{2}{1-\omega^2}$

Solution

a) **Sum of two roots**

$(1 - i\omega^2) + (1 - i\omega)$

$= 2 - i(\omega^2 + \omega)$

$= 2 + i$

Product of two roots

$(1 - i\omega^2)(1 - i\omega)$

$= 1 - i\omega - i\omega^2 + i^2\omega^3$

$= 1 - i(\omega + \omega^2)(-1)(1)$

$= i$

$\therefore$ **the equation is :** $x^2 - (2 + i)x + i = 0$

b) **Sum of two roots**

$$\frac{2}{1-\omega} + \frac{2}{1-\omega^2} = \frac{2 - 2\omega + 2 - 2\omega^2}{1 - \omega^2 - \omega + \omega^3}$$

$$= \frac{4 - 2(\omega + \omega^2)}{2 - (\omega + \omega^2)}$$

$$= \frac{6}{3} = 2$$

Product of two roots

$$\frac{2}{(1-\omega)} \cdot \frac{2}{(1-\omega^2)} = \frac{4}{(1-\omega)(1-\omega^2)}$$

$$\frac{4}{1-\omega^2-\omega+\omega^3} = \frac{4}{1-\omega^2-\omega+1}$$

$$\frac{4}{2-(\omega^2+\omega)} = \frac{4}{2-(-1)} = \frac{4}{3}$$

$\therefore$ **the equation is :** $x^2 - 2x + \frac{4}{3} = 0$

01 COMPLEX NUMBERS

Exercises

1. **Simplify the following expressions ;**

a) ω^{64} b) ω^{-325}

c) $\dfrac{1}{(1+\omega^{-32})^{12}}$ d) $(1+\omega^2)^{-4}$ e) ω^{9n+5} , $n \in w$

2. **Find the quadratic equation whose roots are :**

a) $1+\omega^2$, $1+\omega$ b) $\dfrac{\omega}{2-\omega^2}$, $\dfrac{\omega^2}{2-\omega}$

c) $\dfrac{3i}{\omega^2}$, $\dfrac{-3\omega^2}{i}$

3. **If** $z^2+z+1=0$ **then evaluate** $\dfrac{1+3z^{10}+3z^{11}}{1-3z^7-3z^8}$

4. **prove that :**

a) $\left(\dfrac{1}{2+\omega}-\dfrac{1}{2+\omega^2}\right)^2=-\dfrac{1}{3}$ b) $\dfrac{\omega^{14}+\omega^7-1}{\omega^{10}+\omega^5-1}=\dfrac{2}{3}$

c) $\left(1-\dfrac{2}{\omega^2}+\omega^2\right)\left(1+\omega-\dfrac{5}{\omega}\right)=18$ d) $(1+\omega^2)^3+(1+\omega)^3=-2$

1 – 7 Geometrical Representation of Complex Number

If E^2 or R^2 represents the orthogonal axes Euclidean plane.By corresponding each complex number **x + yi** where x, y ∈ R, at the point (x, y) in E^2, we get a corresponding application from E to R^2. In this plane, we will geometrically represent some of the simple algebraic operation in addition and subtraction on E, which grometrically correspond to operations in E^2 or R^2.

In this section, we will address representation of some operations on geometrically complex numbers, whose shapes will be called Argand shapes (according to the scientist **J. R. Argand, 1768-1822**, the plane is named after the wellknow German scientist **C. F. Gauss, 1777-1855** (Gauss plane) or for short, a complex plane.

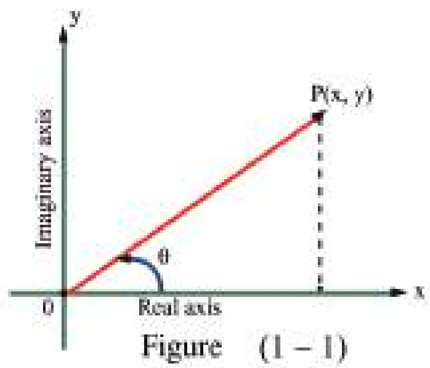


Figure (1 – 1)

The x-axis is called the real axis where it represents the real part of the complex number, as for the y-axis, it is called the imaginary axis, which represents the imaginary part of the complex number. Consequently, the complex number x + yi, is geometrically represented by (x, y), see fig. (1 – 1)

If $z_1 = x_1 + y_1 i$, $z_2 = x_2 + y_2 i$ were two complex numbers represented by points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, then : $z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i$.

Also, $z_1 + z_2$ can be represented by $P_3(x_1 + x_2, y_1 + y_2)$ using information related to the vectors as in fig. (1 – 2)

i.e. $\overrightarrow{0p_1} + \overrightarrow{0p_2} = \overrightarrow{0p_3}$

The complex number x + yi can be represented by vector $\overrightarrow{0p}$, so, the sum of two complex numbers is the sum of two vectors.

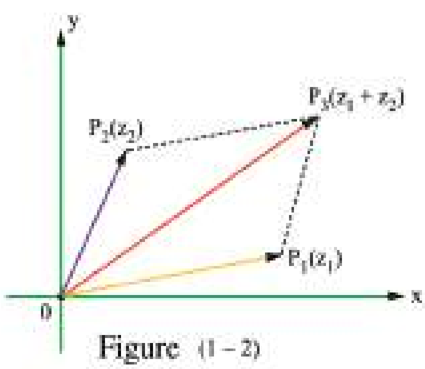


Figure (1 – 2)

Considering $\overrightarrow{P_2}$ represent the complex number $-z_2$, then $\overrightarrow{P_2}$ is the result of $\overrightarrow{0P_2}$ rotation on 0 by half-round this : $z_1 - z_2 = z_1 + (-z_2)$

This is cuopled whit point P_4 whereby $0P_1P_4P_2$ is a similar to a parallelogram $0P_1\ P_3P_2$ as in fig. (1 – 3)

i.e. $\overrightarrow{0P_4} = \overrightarrow{P_2P_1} = \overrightarrow{0P_1} - \overrightarrow{0P_2}$

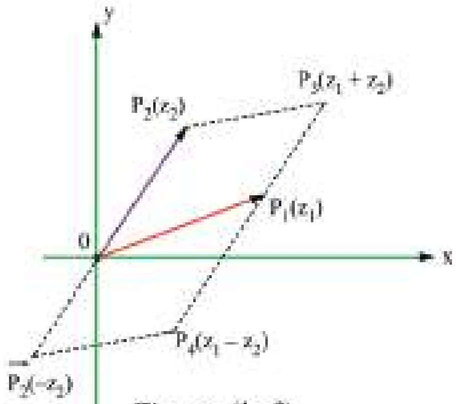


Figure (1 – 3)

Note

1. Let **k** be a non-zero real number. **z** is a complex number, the point represented by **kz** can be obtained by dilation at center **0** and **k** is constant coefficient.
2. Each complex number **z**, the point iz can be obtained from anti-clockwise quadrant period.

Example 22

Represent the following operations geometrically using Argand shape :

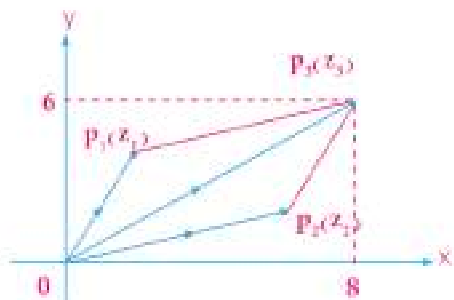
a) $(3 + 4i) + (5 + 2i)$ b) $(6 - 2i) - (2 - 5i)$

Solution

a) $(3 + 4i) + (5 + 2i) = 8 + 6i = P_3(8, 6)$

$z_1 = 3 + 4i \qquad = P_1(3, 4)$

$z_2 = 5 + 2i \qquad = P_2(5, 3)$



Note that : $\overrightarrow{0P_1} + \overrightarrow{0P_2} = \overrightarrow{0P_3}$ is similar to sum of vectors and $0P_1P_3P_2$ is a parallelogram with diagonal. $\overrightarrow{0P_3}$

b) $(6 - 2i) - (2 - 5i) = (6 - 2i) + (-2 + 5i) = 4 + 3i$

$z_1 = 6 - 2i \Rightarrow P_1(z_1) = P_1(6, -2)$

$z_2 = -2 + 5i \Rightarrow P_2(z_2) = P_2(-2, 5)$

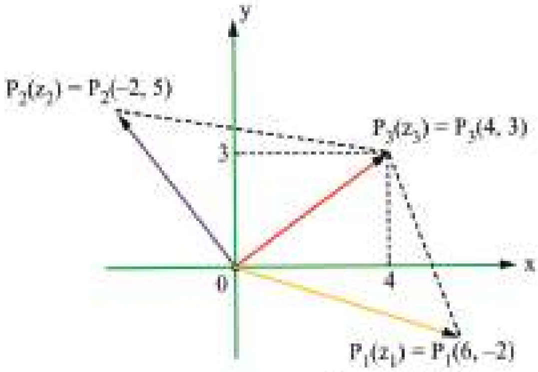


Figure (1 – 5)

$z_3 = 4 + 3i \Rightarrow P_3(z_3) = P_3(4, 3)$

Exercises

1. **Write the additive inverse for the following numbers then represent these numbers and their addivite inverse as Argand shape.**

$z_1 = 2 + 3i \quad , \quad z_2 = -1 + 3i \quad , \quad z_3 = 1 - i \quad , \quad z_4 = i$

2. **Write the conjugate number for each of the following then represent them along with conjugate as Argand shape :**

$z_1 = 5 + 3i \quad , \quad z_2 = -3 + 2i \quad , \quad z_3 = 1 - i \quad , \quad z_4 = -2i$

3. **If $z = 4 + 2i$ then, explain the following in the form of Argand shape.**

$z, \bar{z}, -z$

4. **If $z_1 = 4 - 2i$, $z_2 = 1 + 2i$ then, explain the following in the form of Argand shape each of :**

$-3z_2 , 2z_1 , z_1 - z_2 , z_1 + z_2$

1 – 8 Polar form of Complex Number

In the previous sections, we studied the complex number in algebraic form **z = x + yi** and Cartesian form z = (x, y), in this section; we will study another form of the complex number, which is called polar form and transform one to another.

If we have the complex number **z = x + yi** represented by P(x, y), as in fig. (1 – 6), then (r, θ) are the polar coordinates of the point **P**, where 0 represents pole and $\overrightarrow{0x}$ represents the initial side, this means:

$\theta = m\angle(x0p)$ and $r = \|\overrightarrow{0p}\|$

The measure of θ from $\overrightarrow{0x}$ to $\overrightarrow{0p}$ is anti clockwise if the measure was positive and clockwise if the measure was negative, thus :

$R(z) = x = r\cos\theta$ ...... (1)

$I(z) = y = r\sin\theta$ ...... (2)

Where **R(z)** is the real part of the complex number **z** while **I(z)** is the imaginary part of the complex number

r is called (Modulus of complex number). **z**.

It is a non-negative real number and read as "**mod z**" or **z** Modulus is $\|z\|$ whereby;

$r = \|z\| = \sqrt{x^2 + y^2}$.

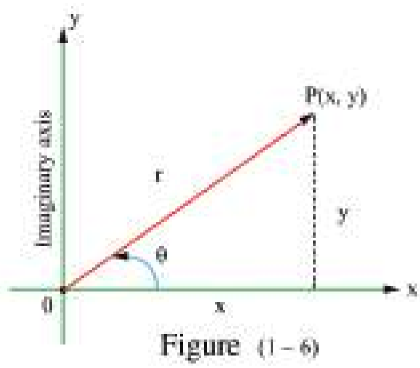


Figure (1 – 6)

From both relations (1) and (2), we get :

$$\cos\theta = \frac{x}{r} = \frac{x}{\|z\|}$$

$$\sin\theta = \frac{y}{r} = \frac{y}{\|z\|}$$

As for θ, its measure is called (**argument of complex number**), written θ = arg(z) for short.

Note

> θ can have infinite number of values, which differ from each other for an integer of rounds.

If θ is argument of a complex number, then both $\theta + 2n\pi$, where n is an integer which becomes an argument too for the same complex number.

As for $\theta \in [0, 2\pi)$ signal of complex number argument, it is called the principle value of the complex number.

Example 23

If $z = 1 + \sqrt{3}\,i$, then find the Modulus and the principle value of z argument.

Solution

$$\text{mod } z = \|z\| = \sqrt{x^2 + y^2} = \sqrt{1+3} = 2$$

$$\cos\theta = \frac{x}{\|z\|} = \frac{1}{2}$$

$\sin\theta = \dfrac{y}{\|z\|} = \dfrac{\sqrt{3}}{2}$ We conclude that θ in the first quadrant

$$\therefore \ \arg(z) = \frac{\pi}{3}$$

Example 24

If $z = -1 - i$, then find the Modulus and the principle value of z argument.

Solution

$$\text{mod } z = \|z\| = \sqrt{1+1} = \sqrt{2}$$

$\cos\theta = \frac{x}{||z||} = \frac{-1}{\sqrt{2}}$

$\sin\theta = \frac{y}{||z||} = \frac{-1}{\sqrt{2}}$ We conclude that θ in the third quadrant

$\therefore \ \arg(z) = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$

Note

1. Argument of complex number **z = 0** is unknown because the zero vector has no direction.

2. Modulus and the principle value of the complex number argument can be used to write the complex number **z = x + yi** in another form called the polar form, as follows :

   $x = r.\cos\theta$, $y = r.\sin\theta$

   $\therefore \ z = r.\cos\theta + i\ r.\sin\theta = r(\cos\theta + i\ \sin\theta)$

   or $z = ||z||(\cos(\arg z) + i\ \sin(\arg z))$

Where $r = \text{mod } z = ||z||$, $\theta = \arg(z)$ is the argument for complex number z.

Example 25

Express each of the following numbers in polar form :

a) $-2 + 2i$ b) $2\sqrt{3} - 2i$

Solution

a) $z = -2 + 2i$

$\text{mod } z = ||z|| = \sqrt{4+4} = 2\sqrt{2}$

$\cos\theta = \frac{-2}{2\sqrt{2}} = \frac{-1}{\sqrt{2}}$

$\sin\theta = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$ θ in the second quadrant

$\arg(z) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

The polar form of the complex number z is :

$z = r(\cos\theta + i\sin\theta)$

$z = 2\sqrt{2}\,(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$

b) $z = 2\sqrt{3} - 2i$

mod $z = \sqrt{12+4} = \sqrt{16} = 4$

$\cos\theta = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$

$\sin\theta = \frac{-2}{4} = \frac{-1}{2}$ θ in the fourth quadrant

$\therefore \arg(z) = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$

The polar form of the complex number :

$z = 4(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6})$

Example 26

Express each of the following numbers in polar form :

a) 1 b) i c) −1 d) − i

Solution

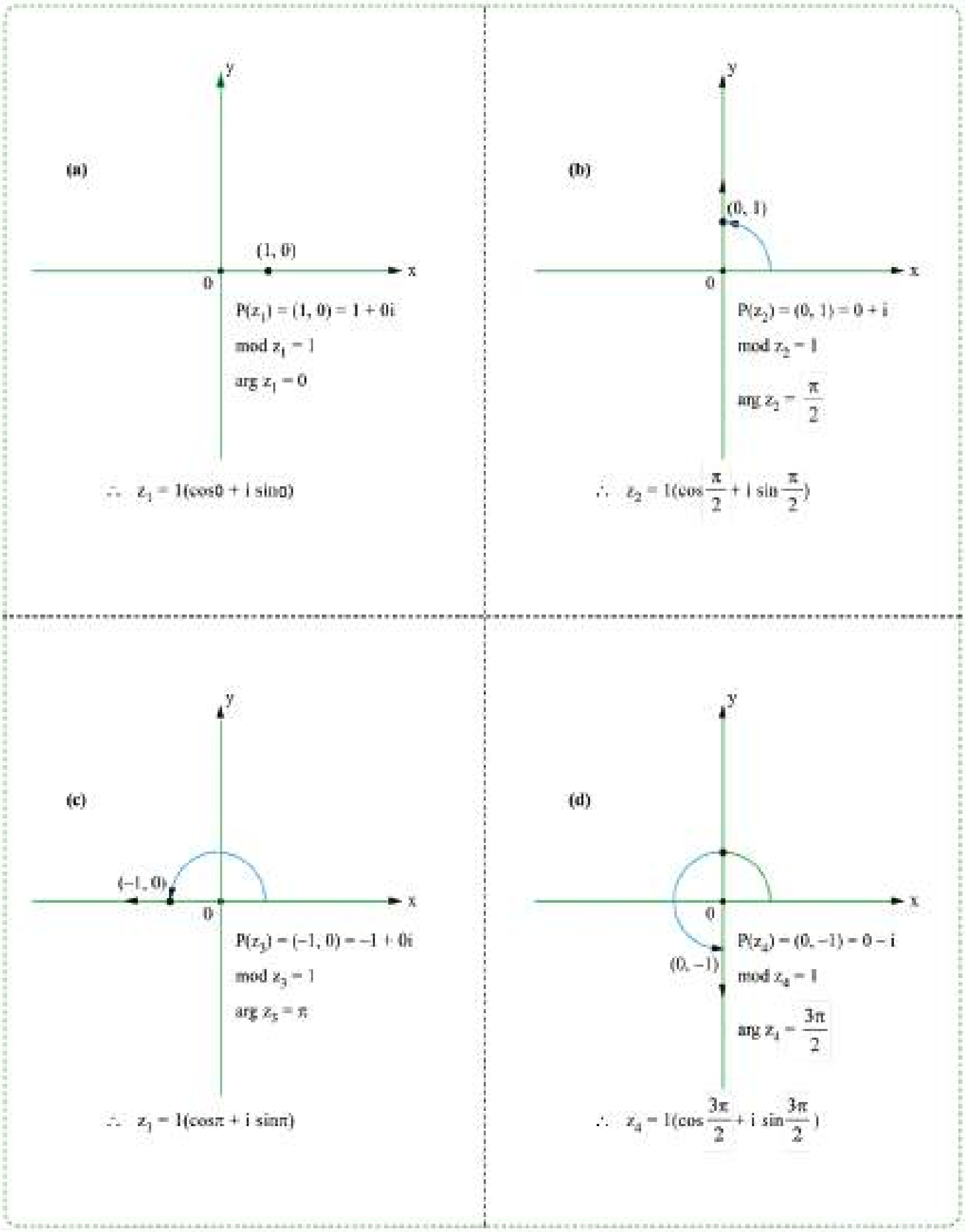


Figure (1 − 7)

Using the previous example, we conclude :

$1 = (\cos 0 + i \sin 0)$

$-1 = (\cos\pi + i \sin\pi)$

$i = (\cos\frac{\pi}{2} + i \sin\frac{\pi}{2})$

$-i = (\cos\frac{3\pi}{2} + i \sin\frac{3\pi}{2})$

Applying the previous conclusion, we can compute :

$3 = 3 \times 1 = 3(\cos 0 + i \sin 0)$

$-2 = 2 \times (-1) = 2(\cos\pi + i \sin\pi)$

$5i = 5 \times i = 5(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2})$

$-7i = 7 \times (-i) = 7(\cos\frac{3\pi}{2} + i \sin\frac{3\pi}{2})$

1 – 9 De Moivre's Theorem

z_2, z_1 can be written as $\mathbf{z_1 = \cos\theta + i \sin\theta}$, $\mathbf{z_2 = \cos\Phi + i \sin\Phi}$

Now we will find $\mathbf{z_1.z_2}$ in polar form

$z_1 \times z_2 = (\cos\theta + i \sin\theta)(\cos\Phi + i \sin\Phi)$

$= \cos\theta.\cos\Phi + i\cos\theta.\sin\Phi + i \sin\theta.\cos\Phi + i^2 \sin\theta.\sin\Phi$

$= [\cos\theta.\cos\Phi - \sin\theta.\sin\Phi] + i [\cos\theta.\sin\Phi + \sin\theta.\cos\Phi]$

$= \cos(\theta + \Phi) + i \sin(\theta + \Phi)$

If ($\Phi = \theta$), then the relation $(\cos\theta + i \sin\theta)^2 = \cos 2\theta + i \sin 2\theta$

It can be proved as follows :

LHS $= (\cos\theta + i \sin\theta)^2 = (\cos^2\theta + 2i \sin\theta\cos\theta - \sin^2\theta)$

$= (\cos^2\theta - \sin^2\theta) + i(2\sin\theta.\cos\theta)$

$= \cos 2\theta + i \sin 2\theta =$ RHS

De Moivre generalized this relation and called De Moivre's Theorem

De Moivre's Theorem

For every $\theta \in R$, $n \in N$, then

$(\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta$

Proof (for information only)

We will proof this theorem by method of mathematical induction, as follows :

1. **Let $n = 1$, then the relation is :**

   $(\cos\theta + i \sin\theta)^1 = \cos 1\theta + i \sin 1\theta$ **which is a true statement.**

2. **Suppose $k \geq 1$ and assume the relation is true for each $n = k$**

   i.e. $(\cos\theta + i \sin\theta)^k = \cos k\theta + i \sin k\theta$ **is hypothetically correct.**

3. **We must prove that the relation is true when $n = k + 1$**

$$\therefore (\cos\theta + i \sin\theta)^{k+1} = (\cos\theta + i \sin\theta)^1 (\cos\theta + i \sin\theta)^k$$
$$= (\cos\theta + i \sin\theta)(\cos k\theta + i \sin k\theta)$$
$$= \cos(\theta + k\theta) + i \sin(\theta + k\theta)$$
$$= \cos(k + 1)\theta + i \sin(k + 1)\theta$$

Thus, if the relation is true at **n**, that is : **$n = k, k \geq 1$**, it is also true at **$n = k + 1$**. This theorem is considered true for all **n** values by using mathematical induction method.

Example 27

Compute : $(\cos\frac{3}{8}\pi + i \sin\frac{3}{8}\pi)^4$

Solution

$$(\cos\frac{3}{8}\pi + i \sin\frac{3}{8}\pi)^4 = \cos\frac{3\pi}{2} + i \sin\frac{3\pi}{2} = 0 + i(-1) = -i$$

Example 28

Show that for each $n \in N, \theta \in R$, then :

$(\cos\theta - i \sin\theta)^n = \cos n\theta - i \sin n\theta$

Solution

LHS $= (\cos\theta - i\sin\theta)^n = [\cos\theta + (-i\sin\theta)]^n$

$= [\cos\theta + i\sin(-\theta)]^n$

$= [\cos(-\theta) + i\sin(-\theta)]^n$

By making, $\phi = -\theta$ the relation becomes :

$= [\cos\phi + i\sin\phi]^n$

$= \cos n\phi + i\sin n\phi$

$= \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta - i\sin n\theta$ RHS

corollary

Corollary of De Moivre's Theorem :

For every $\theta \in \mathbb{R}$, $n \in Z^+$, then

$$\sqrt[n]{z} = r^{\frac{1}{n}}\left[\cos\frac{\theta + 2\pi k}{n} + i\sin\frac{\theta + 2\pi k}{n}\right] \qquad k = 0, 1, 2, \ldots, n-1$$

Example 29

Compute by using De Moivre's Theorem :

$(1 + i)^{11}$

Solution

$z = 1 + i$

$$\text{mod } z = \sqrt{2}, \ \cos\theta = \frac{1}{\sqrt{2}}, \ \sin\theta = \frac{1}{\sqrt{2}} \Rightarrow \therefore \arg z = \frac{\pi}{4}$$

$$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$(1 + i)^{11} = (\sqrt{2})^{11}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{11}$$

$$= 2^{\frac{11}{2}}(\cos\frac{11\pi}{4} + i\sin\frac{11\pi}{4}) = 2^{\frac{11}{2}}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$$

$$= 2^{\frac{11}{2}}(-\frac{1}{\sqrt{2}} + i\,\frac{1}{\sqrt{2}}) = 2^5\sqrt{2}\,(-\frac{1}{\sqrt{2}} + i\,\frac{1}{\sqrt{2}}) = 2^5(-1 + i) = 32(-1 + i)$$

Note

$$(\cos\theta + i\sin\theta)^{-1} = [\cos(-\theta) + i\sin(-\theta)] = (\cos\theta - i\sin\theta)$$

This relation can be generalized as follows :

$$(\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta$$

Example 30

Solve the equation $x^3 + 1 = 0$ **,** $\mathbf{x \in C}$

Solution

$x^3 + 1 = 0 \Rightarrow x^3 = -1$

$x^3 = \cos\pi + i\sin\pi$

$\therefore\ x = (\cos\pi + i\sin\pi)^{\frac{1}{3}}$

$\therefore\ x = \cos\frac{\pi + 2k\pi}{3} + i\sin\frac{\pi + 2k\pi}{3}$

Where k = 0, 1, 2 because it is a cubic root.

Putting $k = 0$, $x_1 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ $\Rightarrow x_1 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

Putting k = 1 , $x_2 = \cos\pi + i\sin\pi \Rightarrow x_2 = -1 + i(0) = -1$

Putting k = 2 $x_3 = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} \Rightarrow x_3 = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

Thus, equation solution set is :

$$\left\{\frac{1}{2} + \frac{\sqrt{3}}{2}i, -1, \frac{1}{2} - \frac{\sqrt{3}}{2}i\right\}$$

Example 31

Find the polar form for :

$$(\sqrt{3}+i)^{\frac{2}{5}}$$

Then find fire roots for it .

Solution

Let $z = \sqrt{3} + i$ **, z is put in the polar form :**

$$||z|| = \sqrt{3+1} = 2$$

$$\cos\theta = \frac{\sqrt{3}}{2} \quad , \sin\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6} \quad , \arg(z) = \frac{\pi}{6}$$

$$z = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) \Rightarrow z^2 = 2^2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})^2$$

$$z^2 = 4(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$$

$$(z^2)^{\frac{1}{5}} = 4^{\frac{1}{5}}[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}]^{\frac{1}{5}} = \sqrt[5]{4}\left[\cos\frac{\frac{\pi}{3}+2k\pi}{5} + i\sin\frac{\frac{\pi}{3}+2k\pi}{5}\right]$$

Whereby k = 0, 1, 2, 3, 4 because it is a fifth root.

$k = 0\ ,\ z_1 = \sqrt[5]{4}(\cos\frac{\pi}{15} + i\sin\frac{\pi}{15})$

$k = 1\ ,\ z_2 = \sqrt[5]{4}\left[\cos\frac{7\pi}{15} + i\sin\frac{7\pi}{15}\right]$

$k = 2\ ,\ z_3 = \sqrt[5]{4}\left[\cos\frac{13\pi}{15} + i\sin\frac{13\pi}{15}\right]$

$k = 3\ ,\ z_4 = \sqrt[5]{4}\left[\cos\frac{19\pi}{15} + i\sin\frac{19\pi}{15}\right]$

$k = 4\ ,\ z_5 = \sqrt[5]{4}\left[\cos\frac{25\pi}{15} + i\sin\frac{25\pi}{15}\right] = \sqrt[5]{4}\left[\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right]$

Exercises

1. Evaluate the following :

a) $\left[\cos\frac{5}{24}\pi + i\sin\frac{5}{24}\pi\right]^4$ b) $\left[\cos\frac{7}{12}\pi + i\sin\frac{7}{12}\pi\right]^{-3}$

2. Evaluate by using De Moivre's Theorem :

a) $(1 - i)^7$ b) $(\sqrt{3} + i)^{-9}$

3. Simplify the following :

a) $\frac{(\cos 2\theta + i\sin 2\theta)^5}{(\cos 3\theta + i\sin 3\theta)^3}$ b) $(\cos\theta + i\sin\theta)^8(\cos\theta - i\sin\theta)^4$

4. Find the square roots the complex number $-1 + \sqrt{3}\,i$ using corollary of De Moivre's Theorem, then the method described in section 1 – 4

5. Using corollary of De Moivre's Theorem, find the cubic roots for the number 27i.

6. Find the four roots for the number(−16) using corollary of De Moivre's Theorem.

7. Find the six roots for the number(−64i) using corollary of De Moivre's Theorem.

CONIC SECTION 02

Chapter 2: Conic Section

2 – 1 Conic Section

2 – 2 Parabola

2 – 3 Translation of the Axes for parabola

2 – 4 Ellipse

2 – 5 Translation of the Axes for Ellipse

2 – 6 Hyperbola

2 – 7 Translation of the Axes for Hyperbola

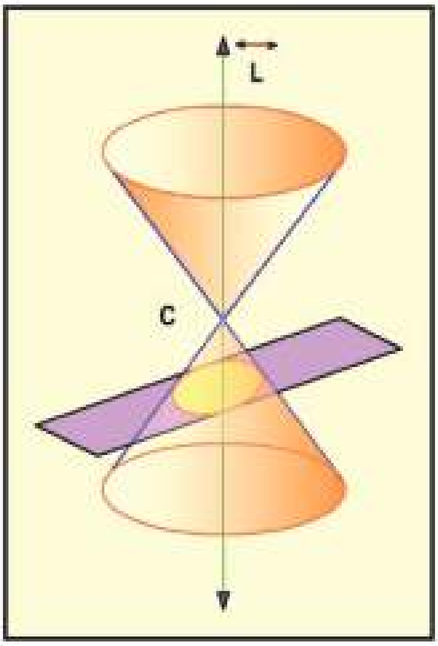
Terminology

Term	Symbol or Mathematical Relation
Focus before Translation	F
Focus after Translation	$\acute{F}$
Eccentricity	$e = \frac{c}{a}$
Constant number	2a

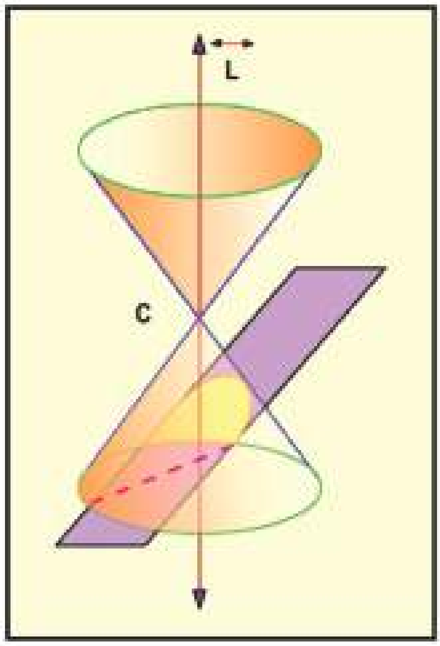
02 CONIC SECTION

2 - 1 Conic Sections

In this section we give geometric definitions of parabolas, ellipses, and hyperbols and derive their standard equations. They are called conic sections. because its consist from intersecting a right circular cone. with a plane as shown in Figure 2 - 1

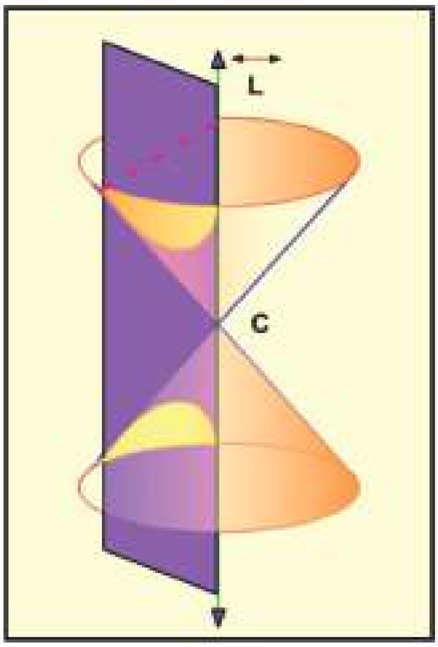


Ellipse



parabola

Figure 2 - 1



Hyperbol

2 - 2 Parabola

Definition : A parabola is the set of points M(x, y) in a plane that are equidistant from a given (fixed) point F(p, 0) and fixed line $\overleftrightarrow{D}$ in the plane.

The fixed point is called **the focus** of the parabola and the fixed line is called the **directrix**.

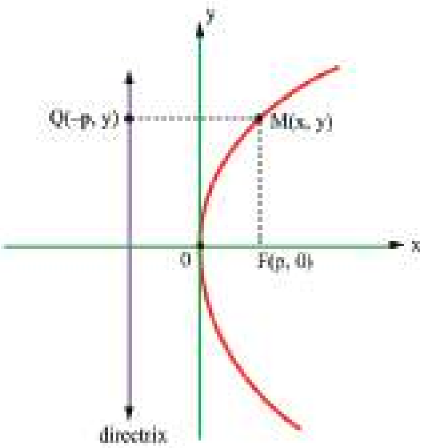
Generally, the focus is shown by the letter "p" such that **p > 0**.

In the given figure (2 – 2)

MF = MQ .... (from definition)

The point "0" is called **vertex** of the parabola and the line which pass through the focus and perpendicular to the directrix pass through it is called **axis of parabola.**

$$\frac{MF}{MQ} = e = 1$$

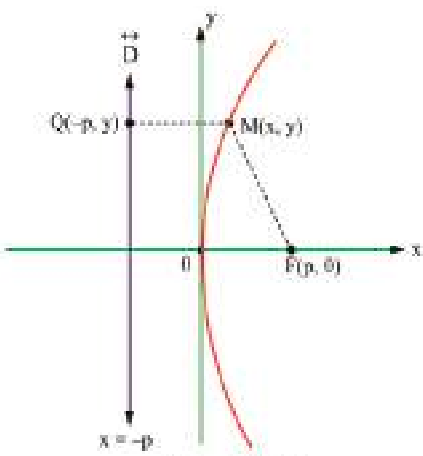


(Figure 2 – 2)

2–2–1 The equation of parabola whose focus is on the x–axis and vertex at the origin

In the coordinate plane, by using definition of parabola we can find the equation of parabola as follow;

Let F(p, 0) be focus of the parabola and the line $\overleftrightarrow{D}$ is the directrix of the parabola, Q(–p, y) is a point on the directrix such that $\overline{MQ} \perp \overleftrightarrow{D}$ and the point M(x, y) is any point on the parabola whose vertex is at origin.



(Figure 2-3 A)
(same Figure 2-4 A)

According to figure 2 – 3 A from the definition of parabola.

MF = MQ

$$\sqrt{(x-p)^2+(y-0)^2} = \sqrt{(x+p)^2+(y-y)^2}$$

$$\sqrt{x^2-2px+p^2+y^2} = \sqrt{x^2+2xp+p^2} \text{ .... (squaring both sides)}$$

$$x^2-2px+p^2+y^2 = x^2+2px+p^2 \text{ .... (simplifying)}$$

$y^2 = 4px\ ,\ \forall p > 0$ .... the equation of parabola.

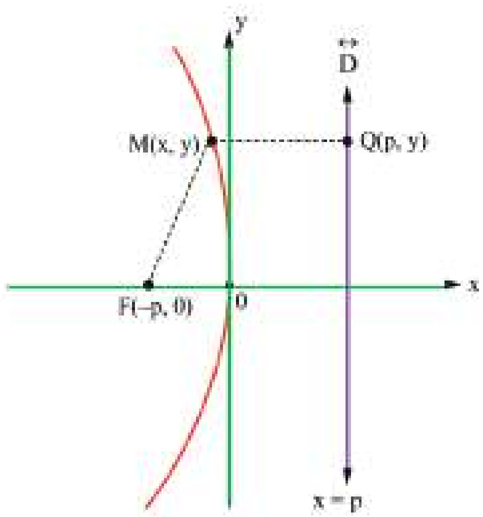
$x = -p$ .... the equation of directrix.

Note

The formulas above belong to the parabola whose focus is on the positive x–axis.

Check Yourself

Try to find equation of parabola whose focus is on the negative x–axis by using definition of parabola in the figure 2 – 3B.



(Figure 2 – 3) B

Example 1

Find the focus and equation of directrix of parabola $y^2 = -8x$.

Solution

$$\left.\begin{aligned} y^2 &= -8x \\ y^2 &= -4px \end{aligned}\right\} \text{ comparing two equations.}$$

$\Rightarrow -4p = -8 \Rightarrow 4p = 8 \Rightarrow p = 2 > 0$

So, $p = 2$

$F(-p, 0) = F(-2, 0)$ ...... focus of parabola.

$x = p$, $x = 2$ .... equation of directrix.

Example 2

Find the equation of parabola knowing that,

a) **Its focus (3, 0) and vertex at the origin.**

b) **The equation of directrix $2x - 6 = 0$ and vertex is at the origin.**

Solution

a) Since $(p, 0) = (3, 0) \Rightarrow p = 3$

$y^2 = 4px$ .... (Since focus is on the positive x–axis)

$\Rightarrow y^2 = (4)\ (3)\ x = 12x$

$\Rightarrow y^2 = 12x$ .... (equation of parabola)

b) from the equation of directrix

$2x - 6 = 0 \Rightarrow 2x = 6 \Rightarrow x = 3$

$p = 3$

$y^2 = -4px$ .... (why?)

$y^2 = (-4)(3)\ x \Rightarrow y^2 = -12x$ .... (equation of parabola)

Example 3

Find the focus and equation of directrix of parabola $y^2 = 4x$ then sketch it.

Solution

$$\left.\begin{array}{l} y^2 = 4px \\ y^2 = 4x \end{array}\right\} \text{comparing two equations.}$$

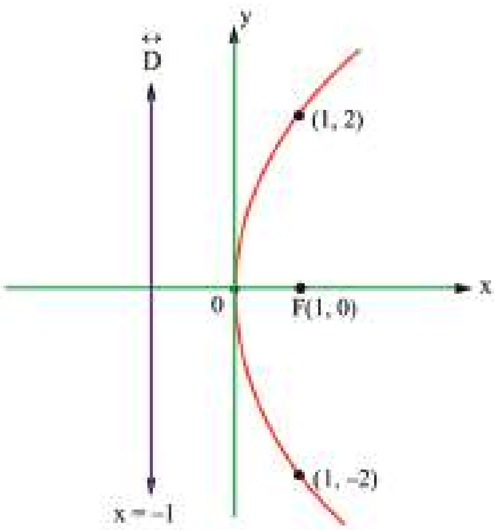
$\Rightarrow 4p = 4 \Rightarrow p = 1$

Focus $F(p, 0) = F(1, 0)$, $x = -1$ ...... equation of directrix

$\mathbf{y^2 = 4x \Rightarrow y = \pm 2\sqrt{x}}$

In order to sketch the graph of parabola we can substitute few points.

x	0	1	2
y	0	± 2	$\pm 2\sqrt{2}$



(Figure 2 – 5)

Conclusion

1

$y^2 = 4px$, $\forall\ p > 0$ .... equation of parabola
$x = -p$ .... equation of directrix
} if the focus is on the positive $x-$axis

$F(p, 0)$

2

$y^2 = -4px$, $\forall\ p > 0$ .... equation of parabola
$x = p$ .... equation of directrix
} if the focus is on the negative $x-$axis

$F(-p, 0)$

Example 4

By using definition of parabola find the equation of parabola whose focus is $F(\sqrt{3}, 0)$ and vertex is at the origin.

Solution

Since $F(\sqrt{3}, 0) = F(p, 0)$

$p = \sqrt{3}$ and equation of directrix is $x = -p \Rightarrow x = -\sqrt{3}$

Let M(x, y) be any point on the parabola and $Q(-\sqrt{3}, y)$ is a point on the directrix ($\overleftrightarrow{D}$) such that

$\overline{MQ} \perp \overleftrightarrow{D}$

From the definition of parabola

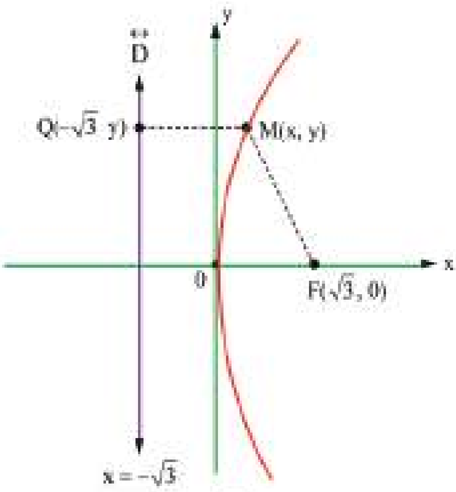
$MF = MQ$

$\sqrt{(x-\sqrt{3})^2+(y-0)^2} = \sqrt{(x+\sqrt{3})^2+(y-y)^2}$ .... (squaring both sides)

$(x-\sqrt{3})^2+y^2 = (x+\sqrt{3})^2$

$x^2-2\sqrt{3}x+3+y^2 = x^2+2\sqrt{3}x+3$ .... (simplifying)

$y^2 = 4\sqrt{3}x$ .... equation of parabola.



(Figure 2 – 6)

2–2–2 The equation of parabola whose focus is on the y–axis and vertex at the origin

In the coordinate plane, by using definition of parabola we can form the equation of parabola as follow;

Let F(0, p) be focus of the parabola and the line $\overleftrightarrow{D}$ is the directrix of the parabola, Q(x, −p) is a point on the directrix such that $\overline{MQ} \perp \overleftrightarrow{D}$ and the point M(x, y) is any point on the parabola whose vertex is at origin. In the figure 2 – 7 – A

According to figure 2 – 7 – A from the definition of parabola.

MF = MQ

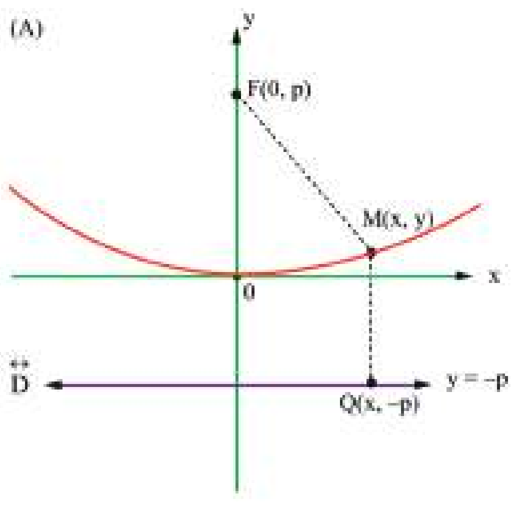
$\sqrt{(x-0)^2+(y-p)^2} = \sqrt{(x-x)^2+(y+p)^2}$ .... (squaring both sides)

$x^2+(y-p)^2=(y+p)^2$

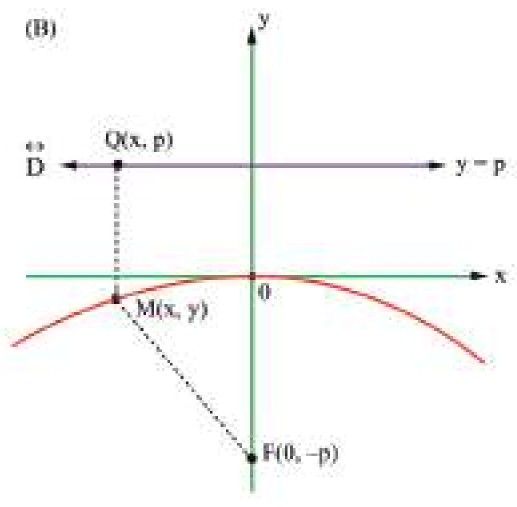
$x^2+y^2-2py+p^2=y^2+2py+p^2$ .... (simplifying)

$x^2 = 4py\ ,\ \forall p > 0$ .... the equation of parabola.

$y = -p$ .... the equation of directrix.



(Figure 2 – 7 – A)



(Figure 2 – 7 – B)

Check Yourself

Try to form equation of parabola whose focus is on the negative y–axis by using definition of parabola in the figure 2 – 7 – B.

Conclusion

1

$x^2 = 4py\ ,\ \forall\ p > 0$ .... equation of parabola
$y = -p$ .... equation of directrix
} if the focus is on the positive y – axis F(0, p)

2

$x^2 = -4py$, $\forall\, p > 0$ .... equation of parabola
$y = p$ .... equation of directrix
} if the focus is on the negative y – axis

$F(0, -p)$

Summary

equation	focus	directrix	axis of parabola	direction
$x^2 = 4py$	$F(0, p)$	$y = -p$	y – axis	open upward
$x^2 = -4py$	$F(0, -p)$	$y = p$	y – axis	open downward
$y^2 = 4px$	$F(p, 0)$	$x = -p$	x – axis	open to the ward right
$y^2 = -4px$	$F(-p, 0)$	$x = p$	x – axis	open to the ward left

Example 5

Find focus and the equation of directrix of parabola $3x^2 - 24y = 0$

Solution

$3x^2 - 24y = 0$ .... dividing both sides by 3.

$\Rightarrow x^2 - 8y = 0$

$\Rightarrow x^2 = 8y$
$x^2 = 4py$
} comparing two equations

$4p = 8 \Rightarrow p = 2 > 0$

$\therefore \quad F(0, p) = F(0, 2)$

$y = -p \Rightarrow y = -2$ .... the equation of directrix.

Example 6

Find the equation of parabola knowing that,

a) **the focus F(0, 5) and vertex is at the origin**

b) **the equation of directrix y = 7 and vertex is at the origin.**

Solution

a) $F(0, p) = F(0, 5)$

$\Rightarrow p = 5$

$x^2 = 4py$ ....(Since focus is on the positive y–axis)

$x^2 = (4)(5)y$

$x^2 = 20y$ ....(the equation of parabola)

b) Since $y = 7$ .... the equation of directrix.

$\therefore p = 7$

$x^2 = -4py$ ....(Since focus is on the negative y–axis)

$x^2 = (-4)(7)y$

$x^2 = -28y$ ....(the equation of parabola)

Example 7

Find the equation of parabola which passes through the points (2, 4) and (2, –4) and vertex is at the origin.

Solution

Since the points (2, 4) , (2, –4) are symmetric with respect to x–axis. So, the equation is

$y^2 = 4px$, $\forall p > 0$

We substitute one of the given points which satisfies the equation of parabola $y^2 = 4px$ for the point (2, 4)

$4^2 = (4)\, p\, (2)$

$16 = 8p \quad \Rightarrow \quad p = 2$

We substitute $p = 2$ in standard equation $y^2 = 4px$

$y^2 = (4)\,(2)\,x$

$\therefore\ y^2 = 8x$ .... the equation of parabola.

Example 8

Find the equation of parabola whose vertex is at the origin and the directrix of parabola passes through the point (3, –5).

Solution

There are two possible cases for the parabola.

I[st] case (focus belongs to x-axis)

$x = 3$ ..... the equation of directrix.

$p = 3$

$y^2 = -4px$

$y^2 = (-4)\,(3)\,x$

$y^2 = -12x$

II[nd] case (focus belongs to y-axis)

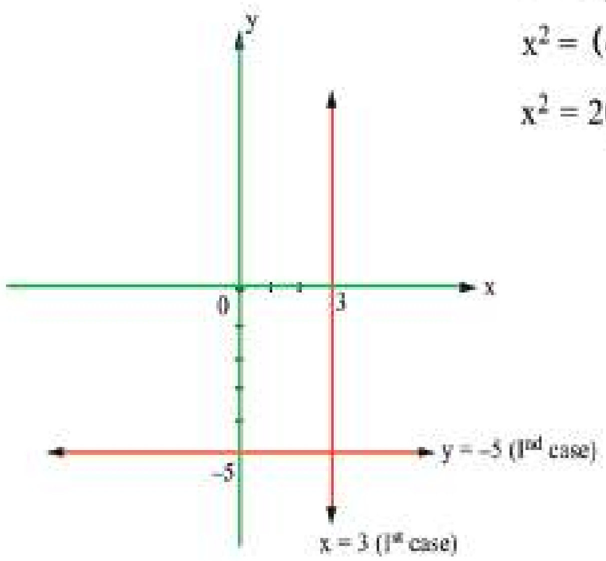
$y = -5$ ..... the equation of directrix.

$p = 5$

$x^2 = 4py$

$x^2 = (4)\,(5)\,y$

$x^2 = 20y$



2–3 Translation of the axes for Parabola

2–3–1 General Equation of The Parabola Whose vertex is at Point (h,k)

In the previous section, we defined two equation of parabola which are

$y^2 = 4px$ ...... (1)

$x^2 = 4py$ ...... (2)

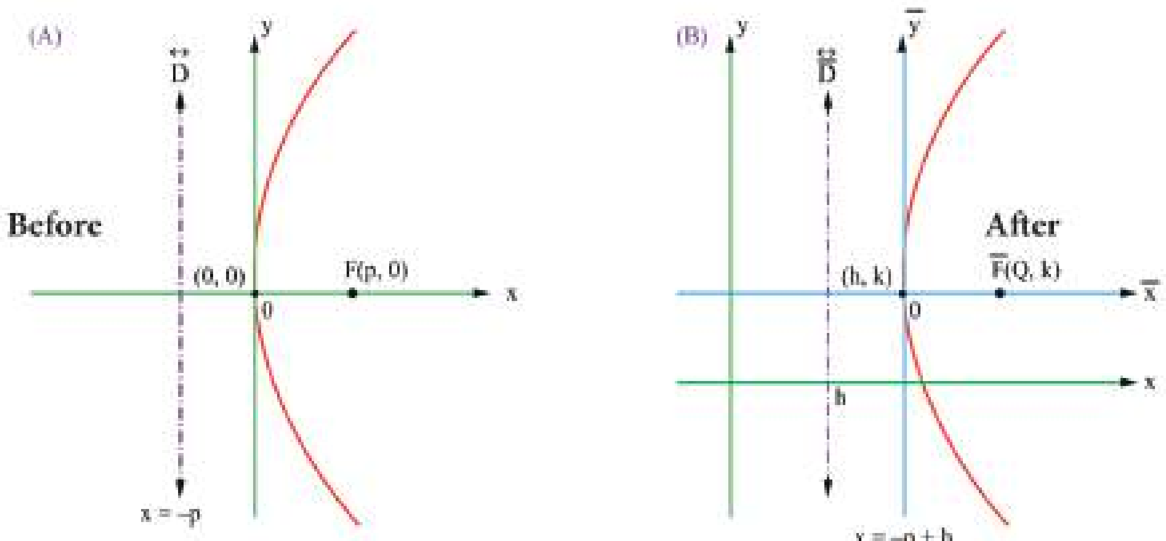
1. Is the equation parabola whose focus is on the x-axis and vertex is at the origin. (0, 0)

2. Is the equation parabola whose focus is on the y-axis and vertex is at the origin. (0, 0)

If the vertex is at point $\overline{O}(h, k)$ then the standard general equations of parabola will be

$(y - k)^2 = 4p(x - h)$ ...... (3)
$(x - h)^2 = 4p(y - k)$ ...... (4)

3. Is the equation of parabola whose vertex is at the point $\overline{O}(h, k)$ and its axis is parallel to x-axis in the Figure 2 – 8.



(Figure 2 – 8)

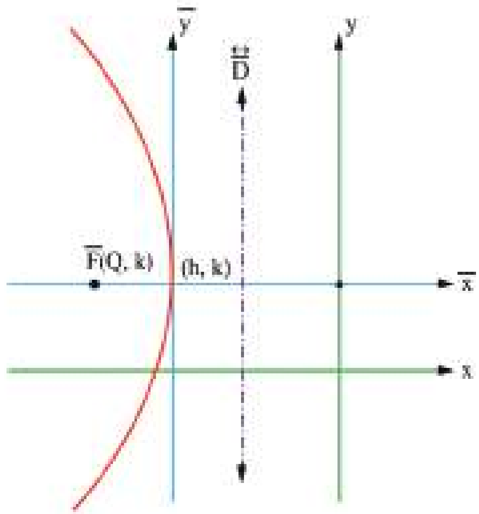
We can conclude the changes of parabola after it is translated.

Vertex	$O(0,0) \rightarrow \overline{O}(h,k)$
Focus	$F(p, 0) \rightarrow \overline{F}(h + p, k)$
Equation of directrix	$x= -p \rightarrow x= -p+h$
Equation of axis of parabola	$y=k$

In the equations (3) and (4) p is the distance between the vertex $\overline{O}$ and focus $\overline{F}$ which is equal to the distance between vertex and equation of directrix such that $p = |Q - h|$.

If the focus of parabola is on the negative x-axis, as in figure 2 - 9 the equations will be as follows,

$(y - k)^2 = - 4p(x - h)$
$\overline{F}(Q, k) = \overline{F}(h - p, k)$
$x = p + h$ ...... equation of directrix
$y = k$ ...... equation of axis of parabola



(Figure 2 – 9)

In the section 2 – 3 (Translation axes) we are going to finding focus and vertex of parabola and equation of directrix, equation of axis of parabola.

Example 9

In the given parabola $(y + 1)^2 = 4(x - 2)$ find the vertex, focus, equation of axis and directrix of parabola.

Solution

$$\left.\begin{array}{l}(y-k)^2 = 4p(x-h)\\ (y+1)^2 = 4(x-2)\end{array}\right\}\text{comparing two equations}$$

$\Rightarrow$ $h = 2$, $k = -1$

the vertex $(h, k) = (2, -1)$

Since $4p = 4 \Rightarrow p = 1$

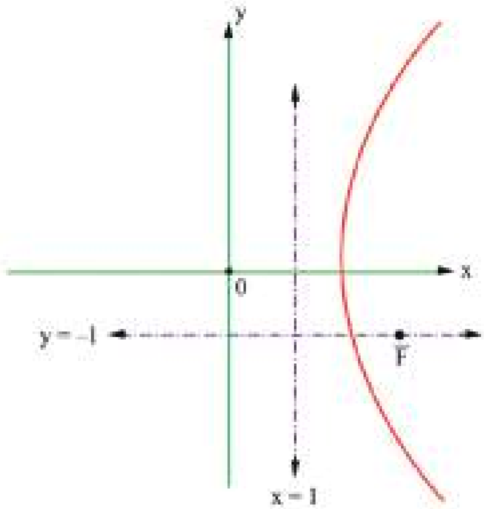
$\overline{F}(p + h, k) = \overline{F}(1 + 2, -1) = \overline{F}(3, -1)$ .... focus

$y = k$

$y = -1$ .... the equation of axis.

$x= -p+h \Rightarrow x = -1 + 2$

$x = 1$ .... the equation of directrix.

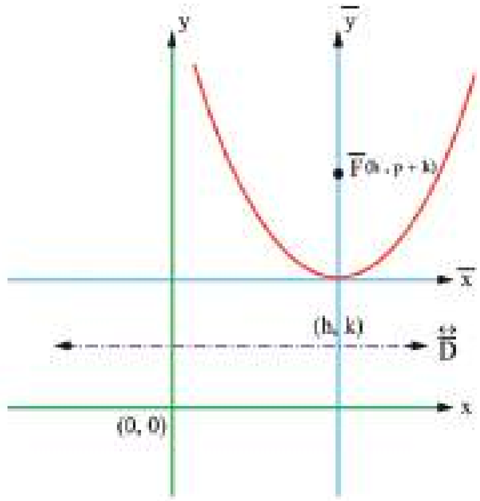


4 is the equation of parabola whose vertex is (h, k) and its axis is parallel to y-axis in the Figure 2–10.

Vertex	$O(0,0) \to \overline{O}(h,k)$
Focus	$F(0, p) \to \overline{F}(h, p+k)$
Equation of directrix	$y= -p \to y= -p+ k$
Equation of axis of parabola	$x=h$

the equation of parabola

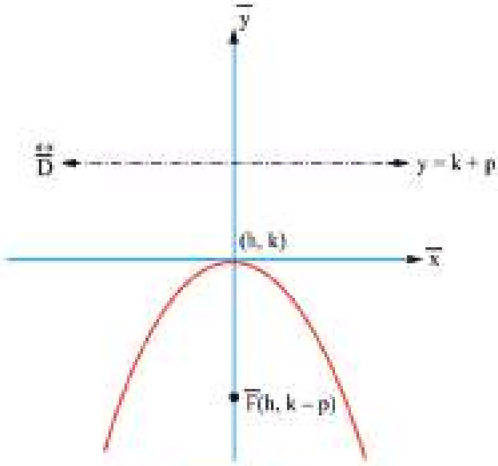
$$(x - h)^2 = 4p(y - k)$$



(Figure 2 – 10)

If the focus of parabola is on the negative y-axis, the equations will be as follow,

$(x-h)^2 = -4p(y-k)$
$\overline{F}(h, -p+k)$ ...... focus
$y = k + p$ ...... equation of directrix
$x = h$ ...... equation of axis of parabola



(Figure 2 - 11)

Example 10

Discuss the parabola $y = x^2 + 4x$

Solution

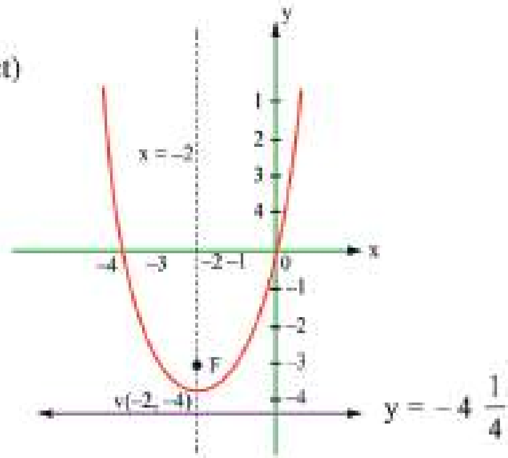
We add 4 to both sides to make right side complete (perfect) square

$y + 4 = x^2 + 4x + 4 = (x+2)^2$

$$\left.\begin{aligned}(x+2)^2 &= y+4\\(x-h)^2 &= 4p(y-k)\end{aligned}\right\}\text{ comparing two equations}$$

$\Rightarrow\quad h = -2\ ,\ k = -4$

the vertex is $(h, k) = (-2, -4) \Rightarrow 4p = 1 \Rightarrow p = \frac{1}{4}$



(Figure 2 – 12)

$\Rightarrow\quad \overline{F}(h, p+k) = \overline{F}(-2, \frac{1}{4} + (-4)) = \overline{F}(-2, -3\frac{3}{4})$ ... focus

$\Rightarrow\quad y = k - p \Rightarrow y = -4 - \frac{1}{4} \Rightarrow y = -4\frac{1}{4}$ ...... equation of directrix.

$\Rightarrow\quad x = h$

$\Rightarrow\quad x = -2$ ...... equation of axis of parabola.

Exercises

1) Find the equation of parabola in each of the following , then sketch the graph of each of them:

a) Focus (5,0) and vertex is at origin .

b) Focus (0,-4) and vertex is at origin .

c) Focus $(0, \sqrt{2})$ and vertex is at origin.

d) Equation of directrix of the parabola is 4y-3=0 and vertex is at origin

2) In the followings , find focus, vertex ,equation of axes ,directrix of parabola:

a) $x^2=4y$ c) $y^2=-4(x-2)$ e) $y^2+4y+2x=-6$

b) $2x+16y^2=0$ d) $(x-1)^2=8(y-1)$ f) $x^2+6x-y=0$

3) Find equation of parabola which passes through the points (2 , -5) , (-2 ,-5), and vertex is at origin .

4) If directrix of parabola passes through the point (-3 ,4) and vertex is at origin , then find its equation when its focus belongs to one of the axes.

5) A parabola whose equation $Ax^2+8y=0$ passes through the point (1 ,2) , find value of A then find focus, directrix and draw the parabola.

6) By using the definition. Find the equation of parabola in each of the followings:

a) Focus (7,0) and vertex is at origin .

b) Equation of directrix $y=\sqrt{3}$ and vertex is at origin.

2–4 ELLIPSE

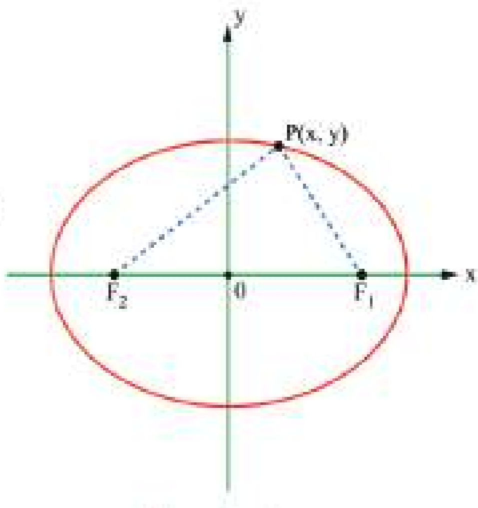
Definition 2-4-1

An ellipse is the set of points in a plane such that the sum of the distances from two fixed points (i.e. two foci) in the plane is a constant number, 2a.

2–4–2 An Ellipse Whose Foci are on x-axis and Center is at The Origin

The foci of ellipse are $F_1(c, 0)$, $F_2(-c, 0)$ and the constant number is 2a, $a > 0$, $c > 0$.

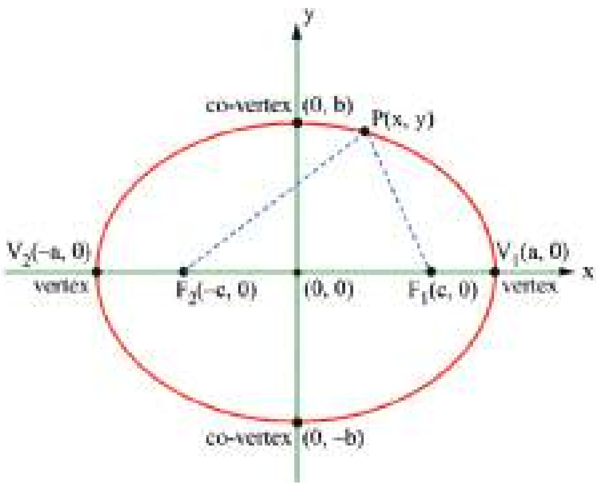
The midpoint of the segment between the foci is called center of ellipse and the line which passes through the foci is called focal axis, the points where this line intersects the ellipse are called vertices of ellipse. The axis which contains the foci of ellipse is called major axis and its length is 2a such that also its equal to the sum of the distance between any point of ellipse P(x, y) and foci of ellipse is



(Figure 2 – 13)

$$PF_1 + PF_2 = 2a$$

The other (small segment) axis is called minor axis and its length is 2b such that $b > 0$ and the end points of minor axis is called "co-vertex" of ellipse.



2–4–3 The Equation of Ellipse Whose Foci are on x–axis and Center at the origin

From Figure 2 - 14

$\therefore$ $PF_1 + PF_2 = 2a$ .... definition of ellipse

$\Rightarrow \sqrt{(x-c)^2+(y-0)^2} + \sqrt{(x+c)^2+(y-0)^2} = 2a$

$\Rightarrow \sqrt{(x-c)^2+y^2} + \sqrt{(x+c)^2+y^2} = 2a$

$\Rightarrow \sqrt{(x-c)^2+y^2} = 2a - \sqrt{(x+c)^2+y^2}$ .... squaring both sides

$\Rightarrow (x-c)^2+y^2 = 4a^2 - 4a\sqrt{(x+c)^2+y^2} + (x+c)^2+y^2$

$\Rightarrow x^2-2cx+c^2+y^2 = 4a^2 - 4a\sqrt{(x+c)^2+y^2} + x^2+2cx+c^2+y^2$

$\Rightarrow 4a\sqrt{(x+c)^2+y^2} = 4a^2+4cx$ ... dividing both sides by 4

$\Rightarrow a\sqrt{(x+c)^2+y^2} = a^2+cx$ ... squaring both sides

$a^2[x^2 + 2cx + c^2 + y^2] = a^4 + 2a^2cx + c^2x^2$

$a^2x^2 + 2a^2cx + a^2c^2 + a^2y^2 = a^4 + 2a^2cx + c^2x^2$ ... simplifying

$a^2x^2 - c^2x^2 + a^2y^2 = a^4 - a^2c^2$

$x^2(a^2 - c^2) + a^2y^2 = a^2(a^2 - c^2)$ .... (1)

and $a^2 = b^2 + c^2$ is main formula for ellipse by substitutig $b^2 = a^2 - c^2$ .... (2) in (1)

$x^2b^2 + a^2y^2 = a^2b^2$ ...... by dividing both sides by a^2b^2

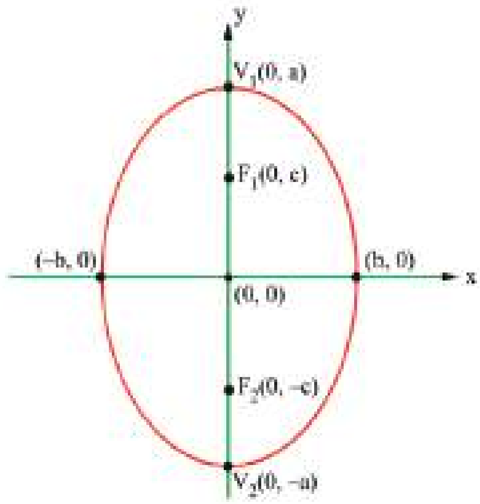
$\frac{x^2b^2}{a^2b^2} + \frac{a^2y^2}{a^2b^2} = \frac{a^2b^2}{a^2b^2} \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ .... standard equation of ellipse.

The value $e = \frac{c}{a}$ is called eccentricity of ellipse and its always less than 1, $e = \frac{c}{a} < 1$

2–4–4 The Equation of Ellipse Whose Foci are on y–axis and Center at the origin

In the Figure 2 - 15

By using same steps of the equation of ellipse whose foci are on x-axis and center is at the origin and by using definition of ellipse we get the equation, $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$ such that; foci are on y-axis and center is at the origin.



(Figure 2 – 15)

Summary

An ellipse whose foci are on x-axis and center is at the origin.	An ellipse whose foci are on y-axis and center is at the origin.
1) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ..... (the equation)	$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$ ..... (the equation)
2) $F_1(c, 0)$, $F_2(-c, 0)$ ..... (foci)	$F_1(0, c)$, $F_2(0, -c)$ ..... (foci)
3) $V_1(a, 0)$, $V_2(-a, 0)$ ..... (vertices)	$V_1(0, a)$, $V_2(0, -a)$ ..... (vertices)

4) $b^2 + c^2 = a^2$

5) $a > c, a > b$

6) the length of major axis is 2a

7) the length of minor axis is 2b

8) the distance between two foci is 2c

9) Area of ellipse $A = ab\,\pi$

10) Perimeter of ellipse $P = 2\pi\sqrt{\frac{a^2+b^2}{2}}$, $\pi = \frac{22}{7}$

11) Eccentricity of ellipse $e = \frac{c}{a}$, $e < 1$, $e = \frac{\sqrt{a^2-b^2}}{a}$

Example 11

For each of the followings, find the length of both axes, foci, vertices and eccentricity.

1) $\frac{x^2}{25}+\frac{y^2}{16}=1$

2) $4x^2+3y^2=\frac{4}{3}$

Solution

1)
$$\left.\begin{array}{l}\frac{x^2}{25}+\frac{y^2}{16}=1\\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ such that } a>b\end{array}\right\}\text{comparing two equations}$$

$\Rightarrow a^2=25 \Rightarrow a=5 \Rightarrow 2a=2(5)=10$ units ..... length of major axis.

$\Rightarrow b^2=16 \Rightarrow b=4 \Rightarrow 2b=2(4)=8$ units ..... the length of minor axis.

$a^2=b^2+c^2$ ..... main formula

$\Rightarrow 5^2=4^2+c^2 \Rightarrow c^2=25-16 \Rightarrow c^2=9 \Rightarrow c=3$

$F_1(3,0)$, $F_2(-3,0)$ ..... foci of ellipse

$V_1(5,0)$, $V_2(-5,0)$ ..... vertices of ellipse

$e=\frac{c}{a}=\frac{3}{5}<1$ ..... eccentricity of ellipse

2)

$4x^2+3y^2=\frac{4}{3}$ ..... multiplying both sides by $\frac{3}{4}$

$$3x^2+\frac{9y^2}{4}=1 \Rightarrow \left.\begin{array}{l}\frac{x^2}{\frac{1}{3}}+\frac{y^2}{\frac{4}{9}}=1\\ \frac{x^2}{b^2}+\frac{y^2}{a^2}=1 \text{ such that } a>b\end{array}\right\}\text{comparing two equations}$$

$a^2=\frac{4}{9} \Rightarrow a=\frac{2}{3} \Rightarrow 2a=\frac{4}{3}$ units ..... the length of major axis.

$b^2=\frac{1}{3} \Rightarrow b=\frac{1}{\sqrt{3}} \Rightarrow 2b=\frac{2}{\sqrt{3}}$ units ..... the length of minor axis.

$a^2 = b^2 + c^2 \Rightarrow \frac{4}{9} = \frac{1}{3} + c^2 \Rightarrow c^2 = \frac{1}{9} \Rightarrow c = \frac{1}{3}$

$F_1(0, \frac{1}{3})$, $F_2(0, -\frac{1}{3})$ ..... foci

$V_1(0, \frac{2}{3})$, $V_2(0, -\frac{2}{3})$ ..... vertices

$e = \frac{c}{a} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2} < 1$ ..... eccentricity

Example 12

Find the equation of ellipse whose foci are $F_1(3, 0)$, $F_2(-3, 0)$ and the vertices are $V_1(5, 0)$, $V_2(-5, 0)$ and center is at the origin.

Solution

Since foci and vertices are on x-axis and center is at the origin,

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\Rightarrow$ $F_1(3, 0)$, $F_2(-3, 0) \Rightarrow c = 3$

$\Rightarrow$ $V_1(5, 0)$, $V_2(-5, 0) \Rightarrow a = 5$

$a^2 = b^2 + c^2$ ..... main formula

$5^2 = b^2 + 3^2 \Rightarrow b^2 = 16 \Rightarrow b = 4$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{25} + \frac{y^2}{16} = 1$ ..... the equation of ellipse

Example 13

Find the equation of ellipse whose center is at the origin and its coordinate are on the two coordinates axes, which intersects x-axis by 8 units and y-axis by 12 units. Then find the distance between two foci and the area of the ellipse's region. (and perimeter)

Solution

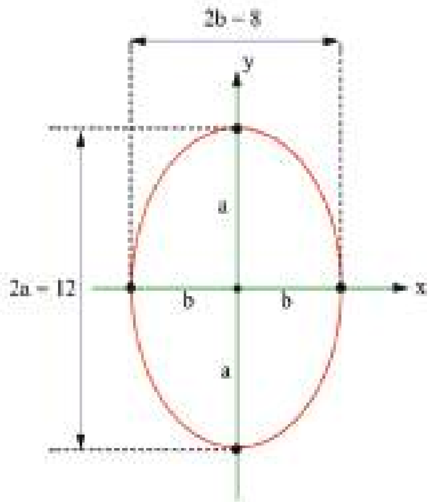
$2b = 8 \Rightarrow b = 4$, $2a = 12 \Rightarrow a = 6$

$\therefore \quad \frac{x^2}{4^2} + \frac{y^2}{6^2} = 1$

$\Rightarrow \frac{x^2}{16} + \frac{y^2}{36} = 1$ ..... the equation of ellipse

$a^2 = b^2 + c^2 \Rightarrow 6^2 = 4^2 + c^2$

$\Rightarrow c^2 = 20 \Rightarrow c = 2\sqrt{5}$



(Figure 2 - 17)

the distance between two foci $\Rightarrow 2c = 4\sqrt{5}$ units.

Area of ellipse $\Rightarrow A = ab\pi$

$\Rightarrow A = (6)(4)\pi \Rightarrow A = 24\pi$ unit square , $\pi = \frac{22}{7}$

Perimeter of ellipse $\Rightarrow P = 2\pi\sqrt{\frac{a^2+b^2}{2}}$

$\Rightarrow P = 2\pi\sqrt{\frac{36+16}{2}} = 2\pi\sqrt{\frac{52}{2}} = 2\sqrt{26}\,\pi$ units

Example 14

Let $kx^2 + 4y^2 = 36$ be the equation of ellipse whose center is at the origin and one of its foci is $(\sqrt{3}, 0)$ then find the value of $k \in \mathbb{R}$.

Solution

$kx^2 + 4y^2 = 36$ .... $(\div 36)$

$\frac{x^2}{\frac{36}{k}} + \frac{y^2}{9} = 1$

From the focus $(\sqrt{3}, 0)$

$\Rightarrow \quad c = \sqrt{3} \Rightarrow c^2 = 3$

$$\left.\begin{array}{l}\dfrac{x^2}{\frac{36}{k}}+\dfrac{y^2}{9}=1\\[2ex] \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1\end{array}\right\}\text{comparing two equations}$$

$\Rightarrow \ a^2=\frac{36}{k}$, $b^2=9$, $c^2=3$ .... (1)

$a^2=b^2+c^2$ .... (2)

by substituting (1) in (2) $\frac{36}{k}=9+3 \Rightarrow \frac{36}{k}=12 \Rightarrow k=3$

Example 15

Find the equation of ellipse whose center is at the origin, foci are on the x-axis, and distance between two foci is 6 units, and difference between the length of the axes is 2 units.

Solution

Since $2c=6 \Rightarrow c=3$, $2a-2b=2 \Rightarrow a-b=1 \Rightarrow a=b+1$ .... (1)

$a^2=b^2+c^2$ .... by substituting (1) in main formula

$(b+1)^2=b^2+c^2 \Rightarrow b^2+2b+1=b^2+c^2 \Rightarrow 2b+1=3^2 \Rightarrow b=4$ .... (2)

by substituting (2) in (1) , $a=b+1 \Rightarrow a=4+1 \Rightarrow a=5$

$\therefore \quad \frac{x^2}{5^2}+\frac{y^2}{4^2}=1 \Rightarrow \frac{x^2}{25}+\frac{y^2}{16}=1$ ..... the equation of ellipse

Example 16

Find the equation of ellipse whose center is at the origin, one of its foci is focus of parabola $y^2-12x=0$ and the length of minor axis is 10 units.

Solution

$y^2 - 12x = 0$

$$\left.\begin{array}{l} y^2 = 12x \\ y^2 = 4px \end{array}\right\} \text{comparing two equations.}$$

$\Rightarrow \quad 12x = 4px \Rightarrow 4p = 12 \Rightarrow p = 3$

$F_1(3, 0)$, $F_2(-3, 0)$ ...... foci of ellipse.

Since $c = 3 \Rightarrow c^2 = 9$

$2b = 10$ .... the length of minor axis.

$\Rightarrow \quad b = 5 \Rightarrow b^2 = 25$

$a^2 = b^2 + c^2$ .... main formula

$\Rightarrow \quad a^2 = 25 + 9 \Rightarrow a^2 = 34$

$\therefore \quad \frac{x^2}{34} + \frac{y^2}{25} = 1$ ..... the equation of ellipse

Example 17

By using the definition of ellipse, find the equation of ellipse whose foci are $F_1(2, 0)$ and $F_2(-2, 0)$ and constant number is 6

Solution

Let P(x, y) be any point on the ellipse.

$PF_1 + PF_2 = 2a$ .... definition of ellipse

$\Rightarrow \sqrt{(x-2)^2 + y^2} + \sqrt{(x+2)^2 + y^2} = 6$

$\Rightarrow \sqrt{(x-2)^2 + y^2} = 6 - \sqrt{(x+2)^2 + y^2}$ .... squaring both sides

$\Rightarrow (x-2)^2 + y^2 = 36 - 12\sqrt{(x+2)^2 + y^2} + (x+2)^2 + y^2$

$\Rightarrow x^2 - 4x + 4 + y^2 = 36 - 12\sqrt{(x+2)^2 + y^2} + x^2 + 4x + 4 + y^2$

$\Rightarrow\ 12\sqrt{(x+2)^2+y^2}=36+8x$ .... dividing both sides by 4.

$\Rightarrow\ 3\sqrt{(x+2)^2+y^2}=9+2x$ .... squaring both sides

$\Rightarrow\ 9(x^2+4x+4+y^2)=81+36x+4x^2$

$\Rightarrow\ 9x^2+36x+36+9y^2=81+36x+4x^2$

$\Rightarrow\ 5x^2+9y^2=45$

$\therefore\ \frac{x^2}{9}+\frac{y^2}{5}=1$ ..... the equation of ellipse

2-4-5 Graph The Ellipse

In order to sketch the graph of ellipse follow the steps,

let $\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$ be the equation of ellipse whose foci are on x-axis.

1) Plot the vertices $V_1(a, 0)$, $V_2(-a, 0)$ on coordinate plane.

2) Plot the co-vertices $M_1(0, b)$, $M_2(0, -b)$

3) Draw the curve which passes through the points $V_1M_1V_2M_2$ respectively.

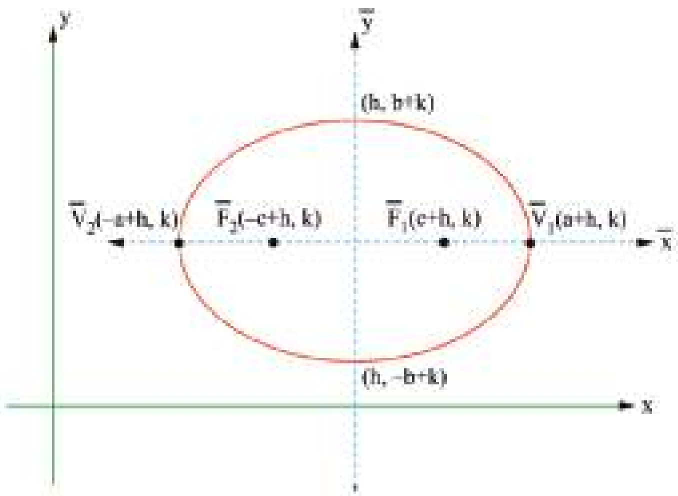
4) Plot the foci $F_1(c, 0)$, $F_2(-c, 0)$

2-5 Transition of the axes for Ellipse

2 –5 –1 The Equation of Ellipse Whose Major Axis is Parallel to the x-axis and Center is (h,k)

When the ellipse is translated from center (0, 0), h units on the x-axis and k units on the y-axis the equation of ellipse will be

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$



(Figure 2 – 19)

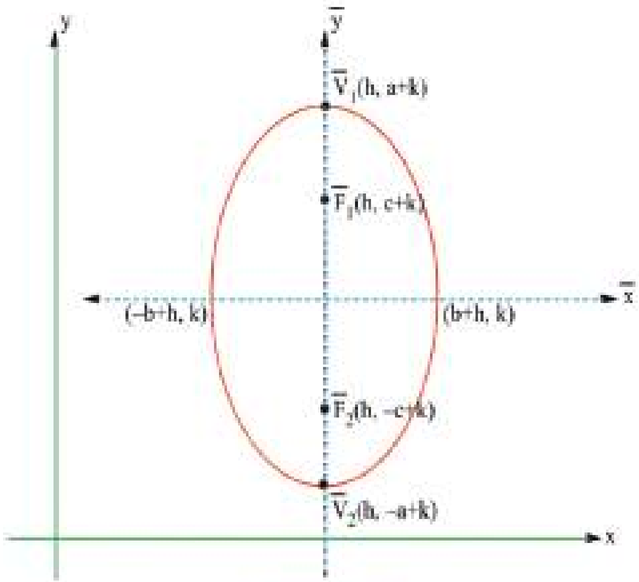
In the figure 2 – 19 the major axis is parallel to x-axis and its length is 2a and its equation y = k. The minor axis is parallel to the y-axis its length is 2b and its equation x = h.

But after translated the ellipse the foci will be $\overline{F_1}$ (c+h, k) , $\overline{F_2}$ (−c+h, k) and the vertices are $\overline{V_1}$(a+h, k) , $\overline{V_2}$(−a+h, k)

2–5–2 The Equation of Ellipse Whose Major Axis is Parallel to the y-axis and Center is (h, k)

When the ellipse is translated from center (0, 0), h units on the x-axis and k units on the y-axis the equation of ellipse will be

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$$



(Figure 2 − 20)

Foci of ellipse are $\overline{F_1}$(h, c+k) , $\overline{F_2}$(h, −c+k) and vertices of ellipse are $\overline{V_1}$(h, a+k) , $\overline{V_2}$(h, −a+k)

The major axis is parallel to y-axis, its length is 2a and its equation is $x = h$.

The minor axis is parallel to x-axis, its length is 2b and its equation is $y = k$.

Note

In the Section 2 – 5 (Translating Ellipse) we are going to focus on only finding center, foci, vertices, (Poles) co-vertices, length of axes and equation of axes of the ellipse.

Example 18

Find foci, vertices, co-vertices, length of axes and equation of axes of ellipse then find the value of e.

$$\frac{(x-2)^2}{9}+\frac{(y-1)^2}{25}=1$$

Solution

$$\left.\begin{aligned}&\frac{(x-2)^2}{9}+\frac{(y-1)^2}{25}=1\\&\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1\end{aligned}\right\}\text{ comparing two equation}$$

$h = 2$, $k = 1 \Rightarrow (h, k) = (2, 1)$ .... center of ellipse

$a^2 = 25 \Rightarrow a = 5 \Rightarrow 2a = 10$ units .... the length of major axis.

$b^2 = 9 \Rightarrow b = 3 \Rightarrow 2b = 6$ units .... the length of minor axis.

$a^2 = b^2 + c^2 \Rightarrow 25 = 9 + c^2 \Rightarrow c^2 = 16 \Rightarrow c = 4$

$$\left.\begin{aligned}&\overline{F_1}(h, c+k)=\overline{F_1}(2, 4+1)=\overline{F_1}(2, 5)\\&\overline{F_2}(h, -c+k)=\overline{F_2}(2, -4+1)=\overline{F_2}(2, -3)\end{aligned}\right\}\text{ foci of ellipse}$$

$$\left.\begin{aligned}&\overline{V_1}(h, a+k)=\overline{V_1}(2, 5+1)=\overline{V_1}(2, 6)\\&\overline{V_2}(h, -a+k)=\overline{V_2}(2, -5+1)=\overline{V_2}(2, -4)\end{aligned}\right\}\text{ vertices of ellipse}$$

$$\left.\begin{aligned}&(-b+h, k)=(-3+2, 1)=(-1, 1)\\&(b+h, k)=(3+2, 1)=(5, 1)\end{aligned}\right\}\text{ co}-\text{vertices of ellipse}$$

$x = 2$ .... equation of major axis.

$y = 1$ .... equation of minor axis.

$e=\frac{c}{a}=\frac{4}{5}<1$ ..... eccentricity of ellipse

02 CONIC SECTION

Exercises

1) Find foci, vertices, poles(co- vertices), and center, then find the length and equation for both axes and .es and eccentricity for the ellipses whose equations below:

a) $x^2+2y^2=1$

b) $9x^2+13y^2=117$

c) $\frac{(x-4)^2}{81}+\frac{(y+1)^2}{25}=1$

d) $\frac{(x+3)^2}{9}+\frac{(y+2)^2}{25}=1$

e) $9x^2+16y^2-72x-96y+144=0$

f) $x^2+25y^2+4x-150y+204=0$

2) Find standard equation for ellipse whose center is at origin point for each of the following:

a) Foci are the points (5,0)and (-5,0),length of major axis is 12 units.

b) Foci are $(0,\pm 2)$ and intersects x−axis at $x=\pm 4$.

c) One of two foci is far away from the ends of the major axis 1 and 5 units, respectively.

d) The eccentricity $=\frac{1}{2}$ and length of minor axis is 12 units.

e) Distance between foci is (8) units, half of minor axis is (3) units.

3) Using the definition find quation of ellipse if:

a) Foci are the points $(0,\pm 2)$, vertices are points $(0,\pm 3)$ and center is at origin.

b) Distance between two foci is (6) units, constant number is (10), foci are on the x-axis and center is at origin.

4) Find the equation of ellipse whose center is at origin, one of two foci is the focus of the parabola whose equation is $y^2+8x=0$, if the ellipse passes through the point $(2\sqrt{3},\sqrt{3})$.

5) Find ellipse equation whose center is at origin, foci are on x-axis, and passes through the points (3,4) , (6,2)

6) Find equation of ellipse whose center is at origin, foci are at intercept point of curve $x^2 + y^2 - 3x = 16$ with y-axis, and it is tangent to the directrix of parabola $y^2 = 12x$.

7) Find equation of ellipse whose foci belong to x-axis, center at origin point, length of major axis is twice the length of minor axis, and intersects the parabola $y^2 + 8x = 0$ at the point whose x-coordinate is -2.

8) An ellipse with equation $hx^2 + ky^2 = 36$, center is at origin point and sum of squares of the length of its axes is 60, one of its foci is the focus of parabola whose equation $y^2 = 4\sqrt{3}x$ what is the value of $h, k \in R$?

9) Find equation of ellipse whose center at origin point and one of its foci is the focus of parabola $x^2 = 24y$, sum of lengths both axes is 36 units.

10) Find equation of ellipse whose foci $F_1(4,0), F_2(-4,0)$ and the point Q belongs to ellipse, such that the perimeter of the triangle QF_1F_2 is 24 units.

2 - 6 HYPERBOLA

Definition 2-6-1

The hyperbola is the set of points in a plane whose such that the absolute value of the difference of the distance from two fixed points (foci) in the plane have a constant number.

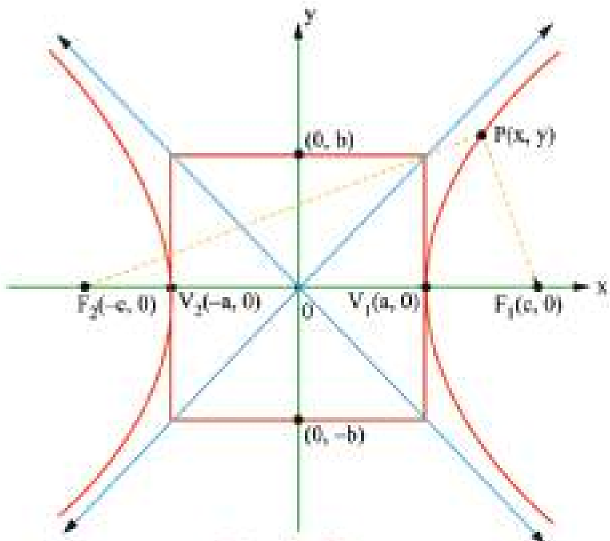
In the figure 2 – 22

$F_1(c, 0)$, $F_2(-c, 0)$ and foci of hyperbola $V_1(a, 0)$, $V_2(-a, 0)$ are vertices of hyperbola and the point $P(x, y)$ is any point on the hyperbola and from the definition

$$|PF_1 - PF_2| = 2a$$

such that 2a is a constant number which represents the length of real axis of hyperbola containing the foci and vertices. The segments PF_1 and PF_2 are called "focal radius" drown from the point P. The distance between two foci F_1, F_2 is equal to 2c.

The length of imaginary axis is 2b.



(Figure 2 – 22)

2–6 –2 The Equation of Hyperbola Whose Foci are on x-axis and Center is at the Origin

From the Figure 2 – 22 and definition of hyperbola,

$\therefore \quad |PF_1 - PF_2| = 2a$

$\Rightarrow \ PF_1 - PF_2 = \pm 2a$

$\Rightarrow \ \sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2} = \pm 2a$

$\Rightarrow \ \sqrt{(x-c)^2 + y^2} = \pm 2a + \sqrt{(x+c)^2 + y^2}$

By squaring and simplifying both sides as we did in the equation of ellipse whose foci are on x-axis and its center is at the origin, we get the equation

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

From Figure 2 – 22 $c > a$, $a > 0$, $c > 0$, $c^2 - a^2 > 0$

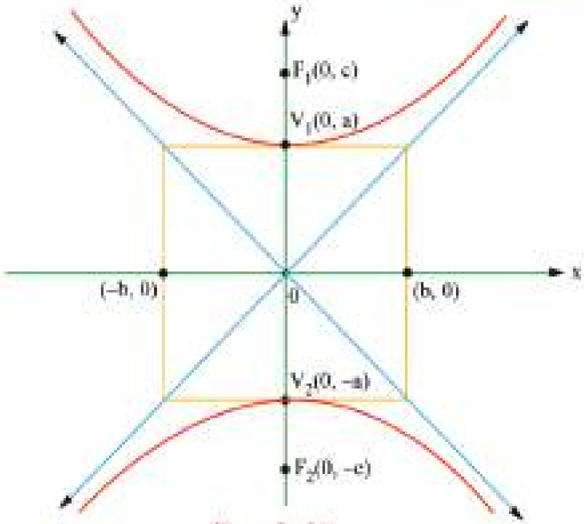
Assuming $b^2 = c^2 - a^2$ and by substituting $a^2 - c^2 = -b^2$ in the equation above, we find that,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$ .... standard equation of hyperbola.

2 – 6 – 3 The Equation of Hyperbola Whose Foci are on y-axis and Center is at the Origin

If foci are on y-axis as shown in the figure 2 – 23 then equation of hyperbola is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$ .... standard equation of hyperbola.



(Figure 2 – 23)

Note

The eccentricity of hyperbola is always greater than 1. $e = \frac{c}{a} > 1$

2 – 6 – 4 Graph The Hyperbola

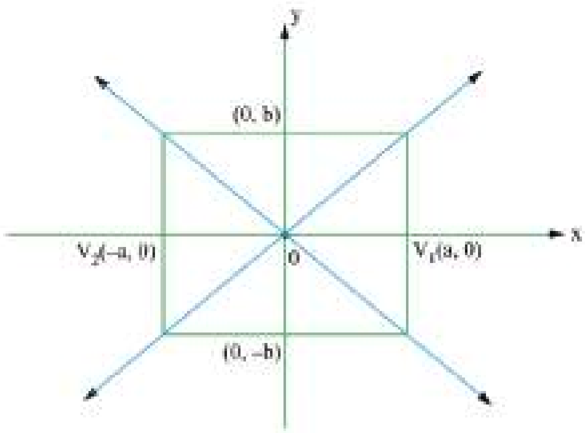
The following steps show how to sketch gragh of hyperbola.

Let $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be the equation of hyperbola whose foci are on x-axis and center is at the origin.

1) Plot the points (a, 0) and (–a, 0) on the coordinate plane.

2) Plot the points (0, b) and (0, –b) on the coordinate plane.

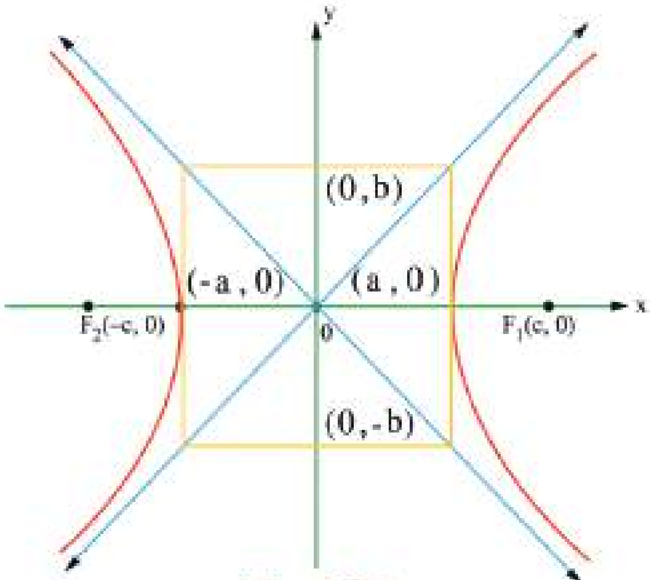
3) From the centeral rectangle which passes through these points as shown in the figure 2 – 24



(Figure 2 – 24)

4) We draw the asymptotes through the diagonals of the central rectangle as shown the Figure 2 –24

5) Plot the foci $F_1(c, 0)$, $F_2(-c, 0)$ then draw the branches of the hyperbola as shown in Figure 2 – 25



(Figure 2 – 25)

Example 19

Find **the foci, vertices, length of axes of hyperbola** $\frac{x^2}{64}-\frac{y^2}{36}=1$ **and sketch it.**

Solution

$$\left.\begin{array}{l}\frac{x^2}{64}-\frac{y^2}{36}=1\\ \frac{x^2}{a^2}-\frac{y^2}{b^2}=1\end{array}\right\}\text{ comparing two equations}$$

$a^2 = 64 \Rightarrow a = 8 \Rightarrow 2a = 16$ units .... the length of real axis.

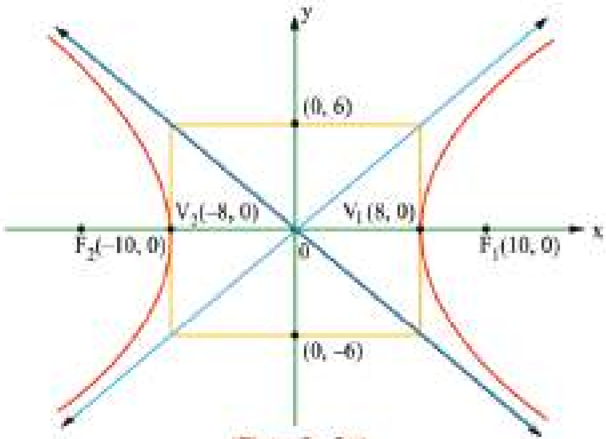
$b^2 = 36 \Rightarrow b = 6 \Rightarrow 2b = 12$ units .... the length of imaginary axis.

$c^2 = a^2 + b^2 \Rightarrow c^2 = 64 + 36 \Rightarrow c^2 = 100 \Rightarrow c = 10$

$$\left.\begin{array}{l} F_1(c, 0) = F_1(10, 0) \\ F_2(-c, 0) = F_2(-10, 0) \end{array}\right\} \text{foci of hyperbola}$$

$$\left.\begin{array}{l} V_1(a, 0) = V_1(8, 0) \\ V_2(-a, 0) = V_2(-8, 0) \end{array}\right\} \text{vertices of hyperbola}$$

$$\left.\begin{array}{l} (0, b) = (0, 6) \\ (0, -b) = (0, -6) \end{array}\right\}$$



(Figure 2 – 26)

Example 20

Find the equation of hyperbola whose center is at the origin, the length of real axis is 6 units and its eccentricity is 2 and foci are on x-axis.

Solution

$2a = 6 \Rightarrow a = 3 \Rightarrow a^2 = 9$

$\therefore \quad e = \frac{c}{a} \Rightarrow 2 = \frac{c}{3} \Rightarrow c = 6$

$\therefore \quad c^2 = a^2 + b^2 \Rightarrow 36 = 9 + b^2 \Rightarrow b^2 = 27$

$\therefore \quad \frac{x^2}{9} - \frac{y^2}{27} = 1$ .... standard equation of hyperbola.

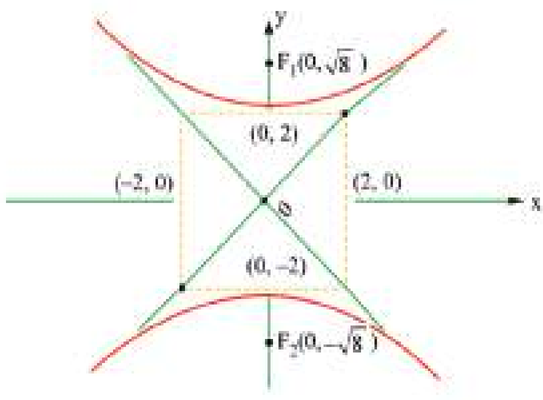
Example 21

Find the equation of hyperbola whose center is at the origin, the length of imaginary axis is 4 units and its foci are $F_1(0, \sqrt{8})$, $F_2(0, -\sqrt{8})$.

Solution

Since the foci are on y-axis the standard equation is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$



(Figure 2 − 27)

$$2b = 4 \Rightarrow b = 2 \Rightarrow b^2 = 4$$

$$c = \sqrt{8}$$

$$\because \quad c^2 = a^2 + b^2 \Rightarrow 8 = a^2 + 4 \Rightarrow a^2 = 4$$

$$\frac{y^2}{4} - \frac{x^2}{4} = 1$$ .... the equation of hyperbola.

In this example, the length of real axis is equel to the length of the coniugate axis (i.e $a^2 = b^2$), tike this kind of hyperbola it is called hyperboliod right angled, because the fore points are formed square shape. The value of the eccentricity equal to $\sqrt{2}$

2-7 Translation of the axes for Hyperbola

2-7-1 The equation of hyperbola whose center is (h, k) and its real axis is parallel to x-axis.

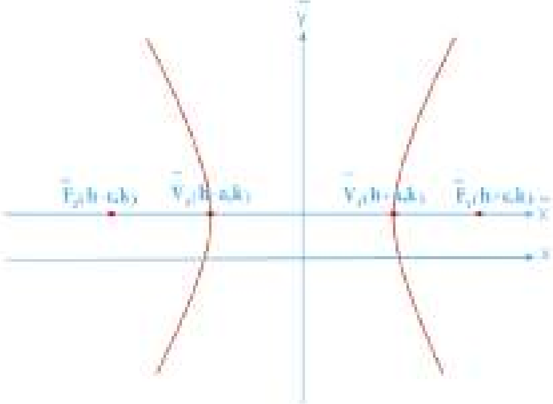
First:

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

As shown in the Figure 2 - 28

Foci of hyperbola are $\overline{F_1}(c+h, k)$, $\overline{F_2}(-c+h, k)$

Vertices of hyperbola are $\overline{V_1}(a+h, k)$, $\overline{V_2}(-a+h, k)$



(Figure 2 - 28)

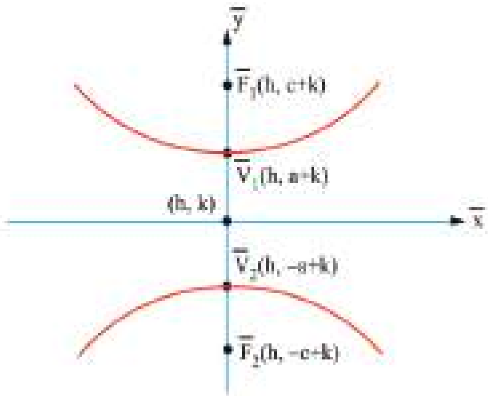
Second : When the hyperbola is translated h units on the x-axis and k units on the y-axis and real axis is parallel to the y-axis then the equation will be,

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$

As shown in the Figure 2 - 29

Foci of hyperbola are $\overline{F_1}(h, c+k)$, $\overline{F_2}(h, -c+k)$

Vertices of hyperbola are $\overline{V_1}(h, a+k)$, $\overline{V_2}(h, -a+k)$



(Figure 2 - 29)

Note

In the section 2–7 we are going to focus on only finding center of hyperbola, foci, vertices and the length of real and imaginary axes.

Example 22

Find the coordinates center, foci, vertices, the length of axes and eccentricity of hyperbola whose equation is

$$\frac{(x+2)^2}{9}-\frac{(y-1)^2}{4}=1$$

Solution

$$\left.\begin{aligned}&\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1\\&\frac{(x+2)^2}{9}-\frac{(y-1)^2}{4}=1\end{aligned}\right\}\text{ comparing two equations.}$$

$a^2=9 \Rightarrow a=3 \Rightarrow 2a=6$ units .... the length of real axis.

$b^2=4 \Rightarrow b=2 \Rightarrow 2b=4$ units .... the length of imaginary axis.

$\Rightarrow h=-2\ ,\ k=1$ center $(h,k)=(-2,1)$

$$c^2=a^2+b^2 \Rightarrow c^2=9+4 \Rightarrow c=\sqrt{13}$$

$$\left.\begin{aligned}\therefore\ &\overline{F_1}(c+h,\,k)=\overline{F_1}(-\sqrt{13}-2,\ 1)\\&\overline{F_2}(-c+h,\,k)=\overline{F_2}(\sqrt{13}-2,\ 1)\end{aligned}\right\}\text{ foci of hyperbola}$$

$$\left.\begin{aligned}\therefore\ &\overline{V_1}(a+h,\,k)=\overline{V_1}(3+(-2),\,1)=\overline{V_1}(1,\,1)\\&\overline{V_2}(-a+h,\,k)=\overline{V_2}(-3+(-2),\,1)=\overline{V_2}(-5,\,1)\end{aligned}\right\}\text{ vertices of hyperbola}$$

$\therefore\quad e=\frac{c}{a}=\frac{\sqrt{13}}{3}>1$ ..... eccentricity of hyperbola.

Exercises

1) Find foci, vertices, length of both axes, and (eccentricity) for following hyperbolas:

a) $12x^2 - 4y^2 = 48$

b) $16x^2 - 9y^2 = 144$

c) $2(y+1)^2 - 4(x-1)^2 = 8$

d) $16x^2 + 160x - 9y^2 + 18y = 185$

2) Write equation of hyperbola in following cases, then graph the hyperbola

a) Foci are points $(\pm 5, 0)$, intercepts with x− axis at $x = \pm 3$, center at origin.

b) Length of real axis 12 units, length of conjugate axis 10 units, both axes congruent on coordinate axes, center at origin.

c) Center at origin, foci at y–axis, length of conjugate axis is $2\sqrt{2}$ units, eccentricity is (3).

3) Find using the definition of hyperbola equation whose center at origin foci are $(2\sqrt{2}, 0)$, $(-2\sqrt{2}, 0)$, its axes congruent on coordinate axes, and absolute value of difference between distances of any point from the foci is (4) units.

4) A hyperbola with real axis (6) units, one of foci is focus of parabola whose vertex is at origin and passes through the points $(1, 2\sqrt{5})$, $(1, -2\sqrt{5})$ Find the equation of parabola whose vertex at origin and hyperbola whose center at origin.

5) A hyperbola with the center at origin, its equation $hx^2 - ky^2 = 90$, length of real axis $6\sqrt{2}$ units, foci congruent on foci of ellipse whose equation is $9x^2 + 16y^2 = 576$. Find value of h,k which belongs to set of real numbers.

6) Write equation of hyperbola whose center at origin if one of its vertices is away from foci by 1 ,9 units respectively, its axes congruent on coordinate axes.

7) Find ellipse equation whose foci are foci of hyperbola whose equation is $x^2 - 3y^2 = 12$, ratio between lengths of axes is $\frac{5}{3}$ and center at origin.

8) The point $P(6, L)$ belongs to hyperbola whose center at the origin and its equation $x^2 - 3y^2 = 12$, find each:

a) L value.

b) Focal radius of hyperbola drawn in the right side of point P.

9) Find hyperbola equation whose foci are ellipse foci $\frac{x^2}{9} + \frac{y^2}{25} = 1$, it tangents directrix of parabola whose equation $x^2 + 12y = 0$

Chater 3 : Application Of Differentiation

1 Related Rates
2 Rolle's and Mean Value Theorems
3 The first derivative test for increasing and decreasing for a function
4 Local Maximum and local Minimum
5 Concavity of curves and inflection point
6 The secand derivatire test Local Maximum and Local Minimum
7 Graphing function
8 Optimization (Maximum, Minimum) problems

Terminology

Team	Symbol or Mathematical Relation
The approximate Change Rate at a	$h\,f'(a)\ ,\ h = b - a$

APPLICATION OF DIFFERENTATION

Prephase: You had studied in fifth class when the function will be differentiable and you had known the rules to find the derivatives of algebraic and circular functions and phsical disriptions.

In this chapter we shall take some concepts and some uses of applications to calculate differentiation.

3 - 1 - Related Rates

Note

Let $y = g(t)$, $x = f(t)$

Where the two variables x , y are depends on variable t

We can find the change rate of each of them as following :

$\frac{dy}{dt} = g'(t)$ Change rate of y with respect to (t)

$\frac{dx}{dt} = f'(t)$ Change rate of x with respect to (t)

For example : The equation

$x^2 + y^2 - 4y + 6x = 0$ we can find the change rate x , y with respect to (t)

$$\frac{d}{dt}(x^2 + y^2 - 4y + 6x) = \frac{d}{dt}(0)$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} - 4\frac{dy}{dt} + 6\frac{dx}{dt} = 0$$

Then: Change rate of y with respect to (t) is $\frac{dy}{dt}$

Change rate of x with respect to (t) is $\frac{dx}{dt}$

Note

In order to solve related rate problems follow the steps the Strategy :

(1) Draw a outline and name the variables and constants, use "t" for the time assume that all variables are diferentiable function of (t) .

(2) Write down any numerical information.

(3) Write down the relation ship that related the variables.

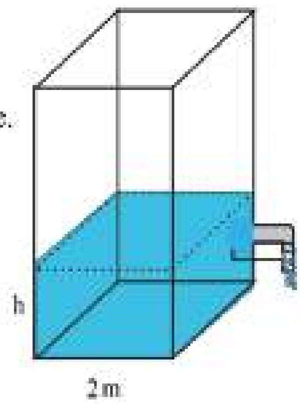
(4) Differentiate both sides of equation with respect to t (time)

(5) Substituting known values in the equation in stap (4)

Example 1

Water is dropping from parallelepiped tank whose base is square with sides 2 meter at a rate of 0.4 m^3/h .

Find rate of change of decreasing water level with respect to time.



Solution

Let the volume of water in parallelepiped be v(t) at time (t), $\frac{dv}{dt} = -0.4$.... (Since water is decreasing, the sign will be negative)

Let the hight of water level in polyhedron be h and we want to find rate of change of height of water level $\frac{dh}{dt}$.

$v = A.h$ (A is the area of the base)

$v = (2).(2).h$

$\Rightarrow v = 4h$ .... differentiate both sides with respect to t

$$\frac{dv}{dt} = 4\frac{dh}{dt}$$

$$-0.4 = 4.\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = -0.1 \text{ m/s}$$

0.1 m/s is the rate of change of decreasing water level.

Example 2

A metal has a shape a rectangle with an area 96 cm^2. If its length increases at a rate of 2 cm/s such that its area will be constant, find the rate of change of decreasing in its width when the width is 8 cm.

Solution

Let x be the length of rectangle and y be the width of rectangle.

rate of change of length is $\frac{dx}{dt} = 2 \text{cm/s}$

rate of change of width is $\frac{dy}{dt} = ?$ when $y = 8$.

$A = xy$

$\therefore 96 = xy$

$\because y = 8 \Rightarrow 96 = x.(8) \Rightarrow x = 12$

$96 = xy$ .... (differentiate both sides) with respect to t

$$\frac{d(96)}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt}$$

$$0 = x.\frac{dy}{dt} + y\frac{dx}{dt}$$

$$0 = 12\frac{dy}{dt} + 8.(2)$$

$$\Rightarrow \frac{dy}{dt} = \frac{-16}{12} = \frac{-4}{3} \text{ cm/s}$$

$\frac{4}{3}$ cm/s is the rate of change of decreasing of the width

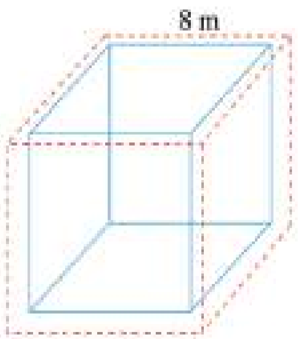
Example 3

A solid cube with aedge 8 cm is covered by a layer of ice. If the ice melts at a rate of $6 \text{ cm}^3 / \text{s}$ and it keeps its shape then find the rate of change of decreasing of ice thickness when thickness is 1 cm

Solution

Let x be the thickness of ice. We want to find $\frac{dx}{dt}$ when $x = 1$.

(Volume of ice) = (Volume of cube with ice) – (Volume of cube)



$v = (8 + 2x)^3 - 8^3$ .... differentiate both sides with respect to (t) .

$$\frac{dv}{dt} = 3(8+2x)^2 . 2 . \frac{dx}{dt} - 0$$

$$-6 = 3\,(8 + 2.(1))^2\,(2)\,\frac{dx}{dt}$$

$$\therefore \frac{dx}{dt} = -0.01 \text{ cm/s}$$

$\therefore$ 0.01 cm/s is rate of change of decreasing of thickness.

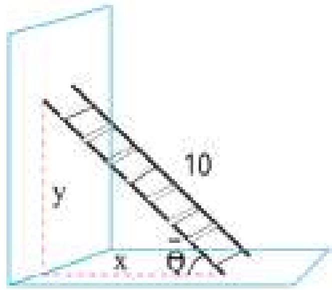
Example 4

A ladder with 10 m length leans vertical a wall. The foot (base) of the ladder is pulled away at a rate of 2 m/s when the foot of the ladder is 8m away from the wall, find.

1. How fast the top of the ladder slides (move) down.

2. Rate of change of angle between the ground and the ladder.

Solution



(1) Let x be the distance between foot of the ladder and wall.

Let y be the distance between top of the ladder and ground.

Let θ be the angle between ladder and ground (rad).

$\frac{dx}{dt} = 2$, find $\frac{dy}{dt}$ when $x = 8$

By using pythagorean theorem, $x^2 + y^2 = 10^2$ with respect to (t) .

if $x = 8 \Rightarrow 8^2 + y^2 = 10^2 \Rightarrow y^2 = 36 \Rightarrow y = 6$

Differentiate both sides of $\Rightarrow x^2 + y^2 = 10^2$

$$\frac{d}{dt}(x^2+y^2)=\frac{d}{dt}(10^2) \Rightarrow 2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$$

By substituting known values in the equation above.

$$(2).(8).(2)+(2).(6).\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{8}{3}\ m/s$$

$\frac{8}{3}$ m/s is rate of change of sliding top of the ladder.

(2) $\sin\theta = \frac{y}{10} \Rightarrow \frac{d}{dt}(\sin\theta) = \frac{d}{dt}\left(\frac{y}{10}\right) \Rightarrow \cos\theta\frac{d\theta}{dt} = \frac{1}{10}\frac{dy}{dt}$

By substituting $\cos\theta = \frac{x}{10} \Rightarrow \frac{x}{10}.\frac{d\theta}{dt} = \frac{1}{10}.\frac{dy}{dt}$

By substituting $x = 8$ and $\frac{dy}{dt} = -\frac{8}{3}$ we get

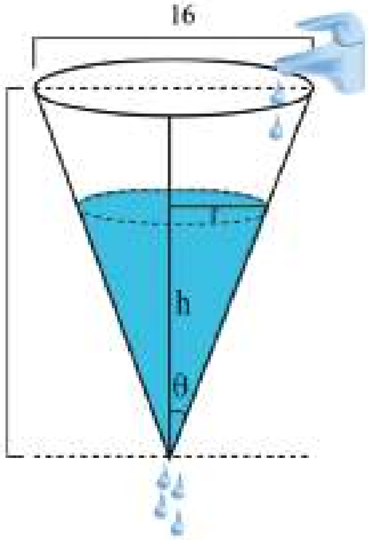
$\frac{8}{10}.\frac{d\theta}{dt} = \frac{1}{10}.\left(-\frac{8}{3}\right) \Rightarrow \therefore \frac{d\theta}{dt} = -\frac{1}{3}$ rad/s is rate of change of angle.

Example 5

A conical filter whose its base is horizontal and its vertex(head) is downward. Its height is 24 cm, and diameter of its base is 16 cm. if the liquid is poured at a rate of 5cm^3 /s, while it leaks liquid at a rate of 1 cm^3/s. Find rate of change of liquid depth when the depth of liquid is 12cm.

Solution

We assume that the dimensions of the liquid at any moment be h (height) and r(radius).



Let the volume of liquid at any moment be v(t).

In the figure at the nearby when we use tanθ or similarity of two triangles, we get

$$\tan\theta = \frac{r}{h} = \frac{8}{24} \Rightarrow r = \frac{1}{3}h$$

$$v = \frac{1}{3}\pi r^2 h \text{.........(substitute } r = \frac{1}{3}h)$$

$$v = \frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 .h = \frac{1}{27}\pi h^3$$

by differentiate both sides with respect to t

$$\therefore \frac{dv}{dt} = \frac{1}{27}.3\pi h^2\frac{dh}{dt} \Rightarrow \frac{dv}{dt} = \frac{1}{9}\pi h^2\frac{dh}{dt}$$

rate of change of volume of liquid = rate of change of pouring liquid – rate of change of leaks liquid

$$\therefore \frac{dv}{dt} = 5 - 1 = 4 \text{ cm}^3/\text{s} \quad \Rightarrow 4 = \frac{1}{9}\pi(12)^2\frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{1}{4\pi}\text{cm/s}$$

Example 6

Let M be a point moving on the parabola curve $y^2 = 4x$ such that rate of change of the point M getting away from the point (7,0) is 0.2 unit/s . Find the rate of change of x-coordinate for M when x=4.

Solution

Let M(x, y) be the point and N(7, 0)

Let the distance between M and N be S.

$$S = \sqrt{(x-7)^2 + (y-0)^2}$$

$$S = \sqrt{x^2 - 14x + 49 + y^2}$$

By substituting $y^2 = 4x$ then,

$$S = \sqrt{x^2 - 14x + 49 + 4x}$$

$$S = \sqrt{x^2 - 10x + 49}$$

Differentiate both sides with respect to (t)

$$\frac{ds}{dt} = \frac{2x-10}{2\sqrt{x^2 - 10x + 49}} \cdot \frac{dx}{dt}$$

$$\Rightarrow 0.2 = \frac{8 - 10}{10} \cdot \frac{dx}{dt}$$

$$\Rightarrow \frac{dx}{dt} = -1 \text{ unit/s}$$

Exercises

1) A ladder leans against a wall. The foot of the ladder is pulled away from the wall at a rate of 2m/s. How fast is the top of the ladder moving down when the angle between the foot of the ladder and ground is $\pi/3$?

2) A pole of length 7.2 m has a light on its top. A man of height 1.8 m is moving away from the pole at a rate of 30 m/min . Find the rate of change in length of man's shadow.

3) Let M be a moving point on the parabola $x^2=y$ .Find the coordinates of M when the rate of change of M getting away from the point (0, 3/2) is two-third of the rate of change in y-coordinate of point M.

4) Find a points which lies on the circle $x^2+y^2+4x-8y=108$ at which the rate of change of x equals the rate of change of y with respect to time ''t ''.

5) polyhedron its dimensions change so that it's base keeps its squared shape, the length of side of the base increases at a rate of 0.3 cm/s and its height decreases at a rate of 0.5 cm/s .Find the rate of change in the volume when the length of side of the base is 4 cm and the height 3 cm.

3 - 2 Rolle`s and Mean Value Theorems

Before beginning to this section.

Definition 3-2-1

If f function is defined on the closed interval [a, b]. then.

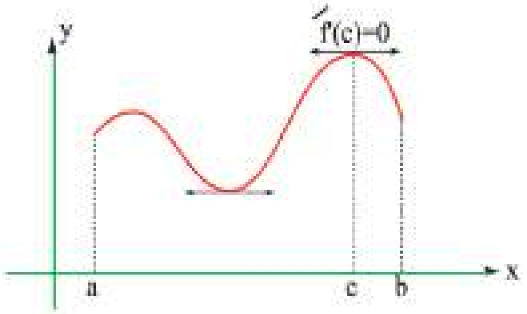
1) f has maximum value at c such that $c \in [a, b]$ if and only if $f(c) \geq f(x)$, $\forall x \in [a, b]$.

2) f has minimum value at c such that $c \in [a, b]$ if and only if $f(c) \leq f(x)$, $\forall x \in [a, b]$.

Theorem 3-2-2

If f function is defined on the closed interval [a, b] and f has maximum or minimum value at c such that $c \in (a, b)$ and $f'(c)$ exits then $f'(c) = 0$

We will explane this theorem geometrically as shown in the figure below.



Example 1

Let $f(x) = |x|$, $f:[-1, 1] \to IR$

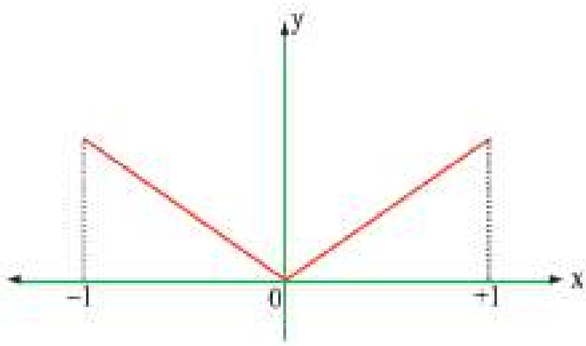
As we see in the figure below the function f has maximum value at each of $x = 1$ and $x = -1$.

And it has minimum value at $x = 0$.

f is not differentiable at $x = 0$

$f'(0)$ doesn't exist.

f doesn't have to satisfy $f'(c) = 0$.



Definition 3-2-3

Let f be defined at c. The number c is called critical number if $f'(c) = 0$ or f is not differentiable at c

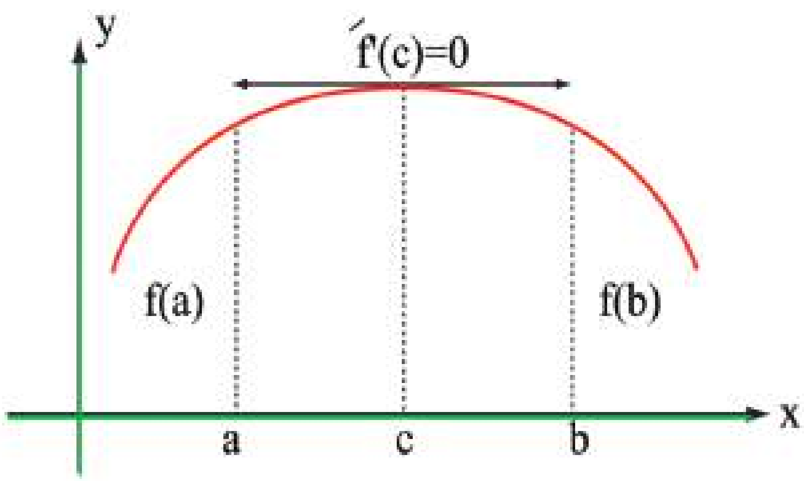
The point (c, f(c)) is called critical point.

In the previous example $f(x) = |x|$, $f : [-1, 1] \to IR$. Notice that is defined at x = 0. But $f'(0)$ doesn't exist.

The number 0 is said to be critical number for the function f. the point (0, f(0)) is critical point.

3-2-4 Rolle`s Theorem

The French scientist M. Rolle made a simple theorem to find the points that represent critical points for the function on the given interval and its named with his name after him.



If f is a function is a

1) continuous on the closed interval [a, b]

2) differentiable on the open interval (a, b)

3) $f(a) = f(b)$

then there exists at least one value of c which belongs to (a, b) and satisfies $f'(c) = 0$.

Example 2

Show that the following functions satisfy Rolle's Theorem for each of the given intervals and find the value of c if exists.

a) $f(x) = (2-x)^2, x \in [0, 4]$

b) $f(x) = 9x + 3x^2 - x^3, \ x \in [-1, 1]$

c) $f(x) = \begin{cases} x^2+1, & x \in [-1,2] \\ -1, & x \in [-4,-1) \end{cases}$

d) $f(x) = k \ , x \in [a, b]$

Note

If the power of the function is natural numbers it is called polynomial function like $7x^2+6x^3+7$

Solution

a) $f(x) = (2-x)^2, x \in [0, 4]$

Ist condition : f is continuous on the interval [0, 4]. Because f is polynomial function.

IInd condition f is differentiable on the interval (0, 4). Because it is polynomial function.

IIIrd condition: $f(0) = (2-0)^2 = 4 \qquad f(4) = (2-4)^2 = 4 \Rightarrow f(0) = f(4)$

f satisfies Rolle's Theorem on the given interval.

$f'(x) = -2(2-x) \Rightarrow f'(c) = -2(2-c)$

$f'(c) = 0 \Rightarrow -2(2-c) = 0 \Rightarrow c = 2 \in (0, 4)$

b) $f(x) = 9x + 3x^2 - x^3 \ x \in [-1, 1]$

Ist condition : f is continuous on the interval [−1, 1]. Because f is polynomial function.

IInd condition : f is differentiable on the interval (−1, 1). Because f is polynomial function.

IIIrd condition : $f(-1) = -9 + 3 + 1 = -5 \qquad f(1) = 9 + 3 - 1 = 11$

$f(-1) \neq f(1)$

f doesn't satisfy Rolle's Theorem on the given interval.

c) $f(x) = \begin{cases} x^2 + 1, & x \in [-1, 2] \\ -1, & x \in [-4, -1) \end{cases}$

The domain $= [-4, 2]$

I^st condition : $\begin{cases} \lim\limits_{x \to -1^+} (x^2 + 1) = 2 = L_1 \\ \lim\limits_{x \to -1^-} (-1) = -1 = L_2 \end{cases}$

$L_1 \neq L_2$ f is not continuous on the interval $[-4, 2]$

f doesn't satisfy Rolle's Theorem on the given interval.

d) $f(x) = k, \quad x \in [a, b]$

I^st condition : f is continuous on the interval $[a, b]$ Because f is constant function.

II^nd condition : f is differentiable on the interval (a, b) Because f is constant function.

III^rd condition : $f(a) = k, \qquad f(b) = k$

$\therefore$ $f(a) = f(b) = k$

$\therefore$ f satisfies Rolle's Theorem on the given interval (a, b)

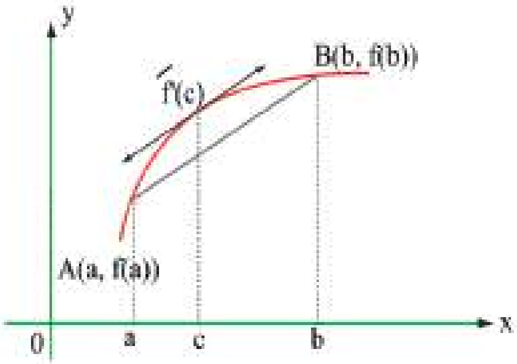
And c value can be any value on (a , b)

3-2-5 The Mean Value Theorem

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) then there exists at least one value c which belongs to (a, b) and satisfies $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ or

$$f(b) - f(a) = f'(c) \cdot (b - a)$$

The following figure explains the Mean Value Theorem geometrically.



The tangent is parallel to chord $\overline{AB}$. Slope of **tangent** is equal to $f'(c)$.

The slope of chord passing through A,B

$$\frac{\triangle y}{\triangle x} = \frac{f(b) - f(a)}{b - a}$$

The slope of tangent to the curve at c = first derivative of f at c.

Therefore, $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Note

The Rolle's Theorem is a special case of the Mean Value Theorem and in Rolle's Theorem 3rd condition $f(a) = f(b)$ must be satisfied.

is : If the chord and tangent parallel to x-axis then $f'(c) = 0$

Example 3

Prove that the following functions satisfy the conditions of the mean value theorem and find the value of c.

a) $f(x) = x^2 - 6x + 4, \quad x \in [-1, 7]$

b) $f(x) = \sqrt{25 - x^2}, \quad x \in [-4, 0]$

Solution

a) $f(x) = x^2 - 6x + 4, \quad x \in [-1, 7]$

1st condition : f is continuous on the interval $[-1, 7]$ Because f is polynomial function.

IInd condition : f is differentiable on the interval (–1, 7) Because f is polynomial function.

The slope of the tangent $f'(x) = 2x - 6 \Rightarrow f'(c) = 2c - 6$

The slope of chord is $\frac{f(b)-f(a)}{b-a} = \frac{f(7)-f(-1)}{7-(-1)} = \frac{11-11}{8} = 0$

Since slope of the tangent is equal to slope of chord.

$\therefore\ 2c - 6 = 0 \ \Rightarrow c = 3 \in (-1, 7)$

b) $f(x) = \sqrt{25-x^2}, \quad x \in [-4, 0]$

Ist condition (continuity) : $\forall a \in [-4, 0] \Rightarrow f(a) = \sqrt{25-a^2} \in IR.$

$\lim_{x \to 4} f(x) = \lim_{x \to 4^+} \sqrt{25-x^2} = \sqrt{25-16} = 3 = f(-4)$ exists.

$\lim_{x \to 0} f(x) = \lim_{x \to 0^-} \sqrt{25-x^2} = \sqrt{25-0} = 5 = f(0)$

$\because \lim_{x \to a} f(x) = f(a) \Leftrightarrow f$ is continuous at a.

Since a is any element of domain $\Leftrightarrow$ f is continuous of [–4, 0].

IInd condition (differentiability): The domain of first derivative of f is

(–5, 5). $\Rightarrow$ f is differentiable $\forall x \in (-4, 0)$

IIIrd condition: Slope of tangent: $f'(x) = \frac{-x}{\sqrt{25-x^2}} \Rightarrow f'(c) = \frac{-c}{\sqrt{25-c^2}}$

Slope of chord: $\frac{f(b)-f(a)}{b-a} = \frac{f(0)-f(-4)}{0-(-4)} = \frac{5-3}{4} = \frac{1}{2}$

Slope of tangent = slope of chord

$$\frac{-c}{\sqrt{25-c^2}} = \frac{1}{2}$$

$$\sqrt{25-c^2} = -2c$$ Squaring both sides

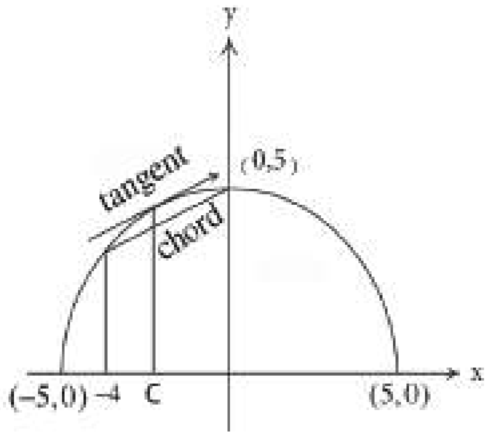
$$25-c^2 = 4c^2$$

$$c^2 = 5$$

$$c = \pm\sqrt{5}$$

$$c = \sqrt{5} \notin (-4,0)$$

$$c = -\sqrt{5} \in (-4,0)$$



Example 4

If $f(x) = x^3 - 4x^2$, $f : [0, b] \to IR$ and f satisfies the mean value theorem at $c = \frac{2}{3}$ then find the value of b.

Solution

$f'(x) = 3x^2 - 8x$ .... (slope of tangent)

$\Rightarrow f'(c) = 3c^2 - 8c$

$\Rightarrow f'\left(\frac{2}{3}\right) = 3\left(\frac{2}{3}\right)^2 - 8.\left(\frac{2}{3}\right) = \frac{4}{3} - \frac{16}{3} = -4$ (slope of tangent)

$\frac{f(b)-f(a)}{b-a} = \frac{f(b)-f(0)}{b-0}$ .... (slope of chord)

$= \frac{b^3 - 4b^2 - 0}{b} = b^2 - 4b$ (Slope of Chord)

slope of chord = slope of tangent

$\therefore\ b^2 - 4b = -4 \Rightarrow b^2 - 4b + 4 = 0 \Rightarrow (b-2)^2 = 0 \Rightarrow b = 2$

3-2-6 Corollary of The Mean Value Theorem

If f is continuous on [a, b] and differentiable on (a, b) and if we consider h = b − a such that $h \neq 0$, $h \in IR$ then by the mean value theorem we get

$f'(c) = \frac{f(b) - f(a)}{b - a}$ .... by substituting b = a + h.

$= \frac{f(a+h) - f(a)}{b - a}$

$\Rightarrow f(a + h) = f(a) + h.f'(c)$

When (b) approached to (a) , the (h) value is small and the chord becomes very small, so the tangent at (c) will be tangent at x = a

So : $f(a + h) \approx f(a) + h f'(a)$

Where: $h f'(a)$ is the approximate change of the function .

Approximation by using corollary of the mean value Theorem

Example 5

Using corollary of mean value theorem to find $\sqrt{26}$ approximately.

Solution

Let $f(x) = \sqrt{x}$ be the function on the interval [25, 26]

$f(x) = \sqrt{x}, \quad \forall x \in [25, 26]$

$f(a) = \sqrt{a}, \ f(25) = \sqrt{25} = 5$

b = 26
a = 25
h = b − a = 1

$f'(x) = \frac{1}{2\sqrt{x}} \qquad f'(25) = \frac{1}{2\sqrt{25}} = \frac{1}{10} = 0.1$

$f(b) \approx f(a) + (b - a) f'(a)$

$f(a + h) \approx f(a) + h f'(a)$

$\sqrt{26} \approx f(25 + 1) \approx f(25) + (1) f'(25)$

$\therefore \sqrt{26} \approx 5 + (1)(0.1) = 5.1$

Example 6

If $f(x) = x^3 + 3x^2 + 4x + 5$. Find f(1.001) approximately using M.V.T

Solution

$b = 1.001$

$a = 1$

$h = b - a = 0.001$

$f(1) = 1 + 3 + 4 + 5 = 13$

$f'(x) = 3x^2 + 6x + 4$

$f'(1) = 3 + 6 + 4 = 13$

$f(a + h) \simeq f(a) + h\, f'(a)$

$f(1.001) \simeq f(1) + (0.001) \,.\, f'(1)$

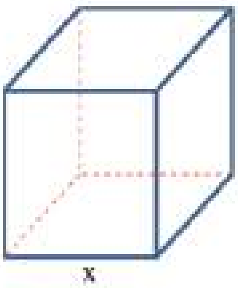
$= 13 + (0.001) \,.\, 13$

$= 13.013$

Example 7

If edge of a cube is 9.98 cm then find the volume of the cube approximately. Using M.V.T

Solution



Let v be the volume of cube

$b = 9.98$

$a = 10$

$h = b - a = -0.02$

$v(x) = x^3 \quad , \quad x \in [9.98, 10]$

$v(10) = 10^3 = 1000$

$v'(x) = 3x^2 \Rightarrow v'(10) = 3\ (10)^2 = 300$

$v(9.98) \cong 1000 + (-0.02)\ (300) \cong 994\ cm^3$

Example 8

Let $f(x) = \sqrt[3]{x^2}$ If x changes from 8 to 8.06 then what the amount of approximate change?

Solution

$f(x) = \sqrt[3]{x^2}$, $f : [8, 8.06] \to IR$

$b = 8.06$

$a = 8$

$h = b - a = 0.06$

$f'(x) = \dfrac{2}{3\sqrt[3]{x}}$

$f'(a) = f'(8) = \dfrac{2}{3\sqrt[3]{8}} = \dfrac{1}{3}$

$hf'(a) \cong (0.06)\ (\frac{1}{3}) = 0.02$ Approximate change

Example 9

A cube with edge 10 cm is painted with 0.15 cm from each edge , find the quantity of paint approximately by using M.V.T

Solution

$v(x) = x^3$

$v'(x) = 3x^2$

$v'(a) = v'(10) = (3)\ (10)^2 = 300$

$b = 10.3$

$a = 10$

$h = b - a = 0.3$

$hv'(10) \simeq (0.3)\ (300) \simeq 90\ cm^3$ volume of the paint approrimately .

Example 10

By using M.V.T find the values of the followings approximately nearest thousand (in decimal part there must be at least 3–digit)

a) $\sqrt[5]{(0.98)^3}+(0.98)^4+3$ b) $\sqrt[3]{7.8}$

c) $\sqrt{17}+\sqrt[4]{17}$ d) $\sqrt[3]{0.12}$

Solution

a) Assume that $f(x)=x^{\frac{3}{5}}+x^4+3$ The function

$f'(x)=\frac{3}{5}x^{\frac{-2}{5}}+4x^3$ Differential both sides

$f(a)=f(1)=1^{\frac{3}{5}}+1^4+3=5$ By substituting a = 1

$f'(a)=f'(1)=\frac{3}{5}.1^{\frac{-2}{5}}+4.\ (1)^3=4.6$

$f(a+h)\cong f(a)+h\,f'(a)$

$b=0.98$
$a=1$
$h=b-a=-0.02$

$f(0.98)=f(1)+(-0.02).(4.6)$

$=5+(-0.02).(4.6)$

$=5-0.092=4.908$

$\sqrt[5]{(0.98)^3}+(0.98)^4+3\cong 4.908$

b) $\sqrt[3]{7.8}$

$b=7.8$
$a=8=2^3$
$h=b-a=-0.2$

Assume that $f(x)=\sqrt[3]{x}$

$f'(x)=\frac{1}{3\sqrt[3]{x^2}}$

$f(a)=f(8)=\sqrt[3]{8}=2$

$f'(a)=f'(8)=\frac{1}{3\sqrt[3]{8^2}}=\frac{1}{12}=0.083$

$f(a+h)\cong f(a)+h\,f'(a)$

$f(7.8) = f(8) + (-0.2)\acute{f}(8)$

$\cong 2 - (0.2)(0.083)$

$= 2 - 0.0166$

$= 1.9834$

$\sqrt[3]{7.8} \simeq 1.9834$

c) $\sqrt{17} + \sqrt[4]{17}$

$b = 17$

$a = 16$

$h = b - a = 1$

Assume that $f(x) = x^{\frac{1}{2}} + x^{\frac{1}{4}}$

$\acute{f}(x) = \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{4}x^{-\frac{3}{4}}$

$f(16) = (2^4)^{\frac{1}{2}} + (2^4)^{\frac{1}{4}} = 4 + 2 = 6$

$\acute{f}(16) = \frac{1}{2}(2^4)^{-\frac{1}{2}} + \frac{1}{4}(2^4)^{-\frac{3}{4}} = \frac{1}{2}(2^{-2}) + \frac{1}{4}(2^{-3}) = 0.5\left(\frac{1}{2}\right)^2 + 0.25\left(\frac{1}{2}\right)^3$

$= (0.5)(0.5)^2 + (0.25)(0.5)^3$

$= (0.5)(0.25) + (0.25)(0.125) = 0.125 + 0.031 = 0.156$

$f(a+h) \simeq f(a) + h\acute{f}(a)$

$f(17) \simeq f(16) + (1)\acute{f}(16)$

$\simeq 6 + (1)(0.156)$

$\sqrt{17} + \sqrt[4]{17} \simeq 6.156$

d) $\sqrt[3]{0.12}$

$b = 0.120$

$a = 0.125$

$h = b - a = -0.005$

Assume that $f(x) = x^{\frac{1}{3}}$

$\acute{f}(x) = \frac{1}{3}x^{\frac{-2}{3}}$

$$f(0.125) = \left((0.5)^3\right)^{\frac{1}{3}} = 0.5$$

$$f'(0.125) = \frac{1}{3}\left[(0.5)^3\right]^{\frac{-2}{3}} = \frac{1}{3}\left(\frac{1}{2}\right)^{-2} = \frac{1}{3}(2)^2 = \frac{4}{3} = 1.333$$

$$f(a+h) \simeq f(a) + h\,f'(a)$$

$$f(0.12) \simeq f(0.125) + (-0.005) \cdot (1.333)$$

$$\simeq 0.5 - 0.006665$$

$$\simeq 0.493335$$

$$\sqrt[3]{0.12} = 0.493335$$

Exercises

1) Find c value defined by Rolle's Theorem in each of the following

a) $f(x) = x^3 - 9x \ , x \in [-3, 3]$

b) $f(x) = 2x + \frac{2}{x} \ , x \in \left[\frac{1}{2}, 2\right]$

c) $f(x) = (x^2 - 3)^2 , x \in [-1, 1]$

2) Find an approximation for each of the following , using mean value theorem :

a) $\sqrt{63} + \sqrt[3]{63}$ b) $(1.04)^3 + 3(1.04)^4$ c) $\frac{1}{\sqrt[3]{9}}$

d) $\frac{1}{101}$ e) $\sqrt{\frac{1}{2}}$

3) A sphere of 6 cm radius, painted with 0.1 cm paint.Find quantity of paint approximately using mean value theorem.

4) A sphere its volume is 84π cm^3 find its radius approximately using mean value theorem .

5) A right circular cone, its height equals base diameter. If height is 2.98cm , then find volume approximately using mean value theorem.

6) Show that each of the following functions satisfy Rolle's Theorem on the given interval for each , then find c value

a) $f(x)=(x-1)^4$, $[-1,3]$
b) $h(x)= x^3- x$, $[-1,1]$
c) $g(x)= x^2- 3x$, $[-1,4]$
d) $f(x) = \cos 2x + 2\cos x$, $[0,2\pi)$

7) Test whether the mean value theorem can be applied on the following functions on the given intervals and mention reasons for that. If the theorem is satisfied, then find possible c values.

a) $f(x) = x^3 - x - 1$, $[-1,2]$
b) $h(x) = x^2 - 4x + 5$, $[-1,5]$
c) $g(x) = \dfrac{4}{x+2}$, $[-1,2]$
d) $b(x) = \sqrt[3]{(x+1)^2}$, $[-2,7)$

3 - 3 The First derivative test for increasing and decreasing for a function

3-3-1 Corollary :

One of the important conclusion of the Mean Value Theorem is following corollary .

Let f be continous on the interval [a, b] and differentiable on the interval (a, b) and if

1) $f'(x) > 0, \quad \forall x \in (a, b) \Rightarrow$ f is increasing function.

2) $f'(x) < 0, \quad \forall x \in (a, b) \Rightarrow$ f is decreasing function.

In this section we will not interest the other case ($f'(x) = 0$)

Example 1

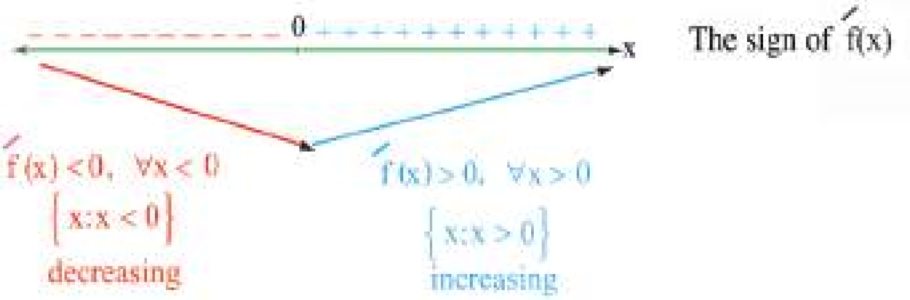
Let $y = f(x) = x^2$. Find the intervals of increasing and deacreasing.

Solution

$y = f(x) = x^2 \Rightarrow y' = 2x$

$y' = 0 \Rightarrow y' = 2.x \Rightarrow 0 = 2x \Rightarrow x = 0$

$y' = 2x$



Example 2

Find the intervals of increasing and decreasing for the following functions :

a) $f(x) = 9x + 3x^2 - x^3$ b) $f(x) = \sqrt[3]{x^2}$

Solution

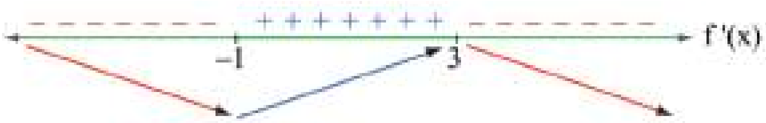
a) $f(x) = 9x + 3x^2 - x^3 \Rightarrow f'(x) = 9 + 6x - 3x^2$

$0 = 9 + 6x - 3x^2$

$0 = -3(x^2 - 2x - 3)$

$0 = (x - 3)(x + 1) \Rightarrow x = 3,\ x = -1$

We examine the sign $f'(x)$ by substituting the values around these numbers on the number line.



$\{x : x < -1\}$, $\{x : x > 3\}$ decreasing function

on the open interval $(-1, 3)$ increasing function.

b) $f(x) = \sqrt[3]{x^2} \Rightarrow f'(x) = \dfrac{2}{3\sqrt[3]{x}}$

$f'(x)$ is undefined at $x = 0$ thus $x = 0$ is critical number.

$\{x : x < 0\}$ decreasing function

$\{x : x > 0\}$ increasing function

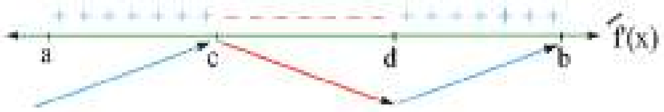
3 - 4 Local maximum and local minimum (Local extremum)

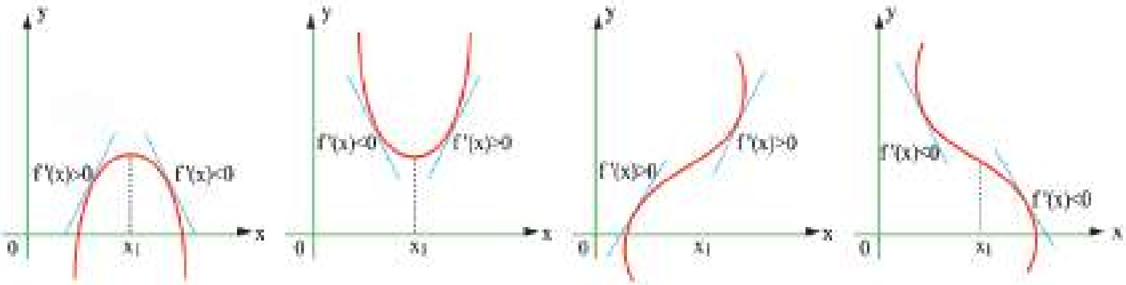
In the figure below the function $y = f(x)$ is increasing on the interval (a, c) since $f'(x) > 0$, it is decreasing on the interval (c, d) since $f'(x) < 0$. And it is increasing on (d, b)

$f'(x) = 0$ at each of $x = c$ and $x = d$.

The point $(c, f(c))$ is called the local maximum point and $f(c)$ is local maximum value of f.

The point $(d, f(d))$ is called the local minimum point and $f(d)$ is local minimum value of f.





the local maximum point   the local minimum point   there doesn't exist local maximum or minimum.

Definition 3-4-1

Let f be continuous on the interval $[a, b]$ and differentiable at $x = c$ which belongs to the interval (a, b) if.

1) $f'(x) < 0; \forall x \in (c, b)$

$f'(x) > 0; \forall x \in (a, c)$

$f'(c) = 0$

then f(c) is the local maximum value

2) $f'(x) > 0; \forall x \in (c, b)$

$f'(x) < 0; \forall x \in (a, c)$

$f'(c) = 0$

then f(c) is the local minimum value

Note

In order to determine the local maximum and local minimum of the function f using the first derivative test follow the steps.

- Find the critical number by solving the equation $f'(x) = 0$.
- Let c be a critical number of function f.
- If sign of $f'(x)$ changes from positive to negative at c then f(x) has local maximum value at c.
- If sign of $f'(x)$ changes from negative to positive at c then f(x) has local mimium value at c.
- If sign of $f'(x)$ does not change at c then f(x) has no local maximum or local minimum value at c.

Example 3

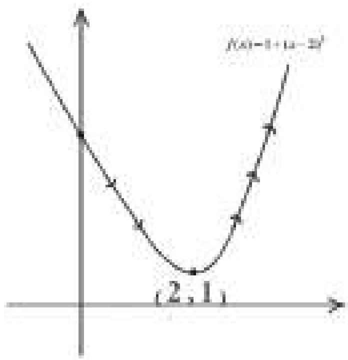
Find the local maximum or minimum point of the following function f.

a) $f(x) = 1 + (x - 2)^2$

b) $f(x) = 1 - (x - 2)^2$

c) $f(x) = x^3 - 9x^2 + 24x$

ing



Solution

a) $f(x) = 1 + (x - 2)^2$

$\Rightarrow f'(x) = 2(x - 2)$

$f'(x) = 0 \Rightarrow 2(x - 2) = 0 \Rightarrow x = 2$

$f(2) = 1 + (2 - 2)^2 = 1$

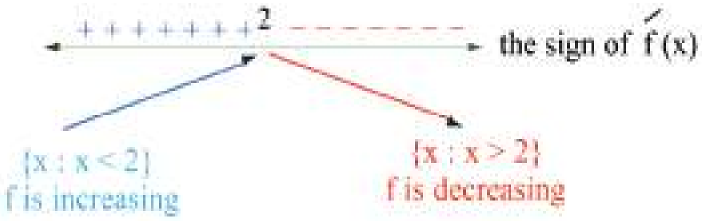
the point (2, f(2)) = (2, 1) represents the local minimum point .

b) $f(x) = 1 - (x-2)^2$

$f'(x) = -2(x-2)$

$f'(x) = 0 \Rightarrow -2(x-2) = 0 \Rightarrow x = 2$

$f(2) = 1 - (2-2)^2 = 1$



The point $(2, f(2)) = (2, 1)$ represents the local maximum.

c) $f(x) = x^3 - 9x^2 + 24x$

$\Rightarrow f'(x) = 3x^2 - 18x + 24$

$f'(x) = 0$

$\Rightarrow 3x^2 - 18x + 24 = 0$

$3(x^2 - 6x + 8) = 0$

$3(x-4)(x-2) = 0 \Rightarrow x = 4 , x = 2$

$f(4) = 16 , f(2) = 20$

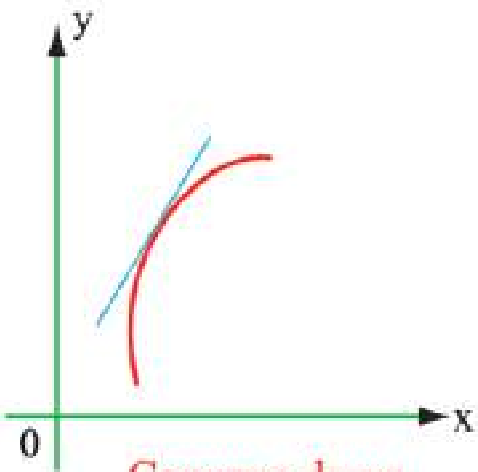
$\{x : x < 2\}, \{x : x > 4\}$ f is increasing

on the open interval (2, 4) f is decreasing.

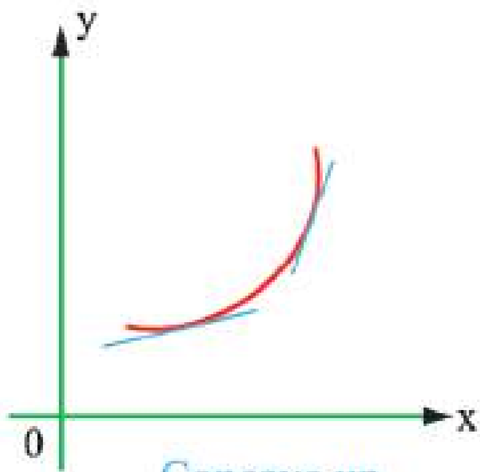
(2, 20) is local maximum point.

(4, 16) is local minimum point.

3 - 5 Concavity and inflection point



Concave down
$f''(x) < 0$
(Figure B)



Concave up
$f''(x) > 0$
(Figure A)

Definition 3-5-1

If f is differentiable on (a, b) and it is said to be f is concave down. If f' is decrasing on the given interval and it is concave up if f' is incrasing on the given interval.

Note

The curve is over the tangent lines on (a, b) ⇔ The curve is concave up

The curve is under the tangent lines on (a, b) ⇔ The curve is concave down as shown in the figure A, B

Theorem 3-5-2

If f is difined on [a, b] and f is twice differentiable on (a, b) then f is concave up if

$\forall x \in (a,b),\ f''(x) > 0$ 
concave up

and f is cancave down on (a, b) if f satisfies

$\forall x \in (a,b),\ f''(x) < 0$ 
concave down

Example 1

Examine the concavity of the following functions.

a) $f(x) = x^2$ b) $f(x) = x^3$

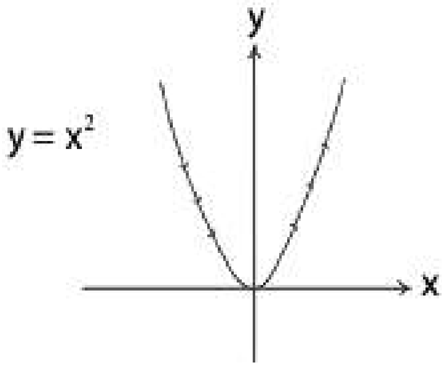
Solution

a) $f(x) = x^2$

$f'(x) = 2x$

$f''(x) = 2$ $\therefore f''(x) > 0,\ \forall x \in IR$

f is concave up on IR



b) $f(x) = x^3$

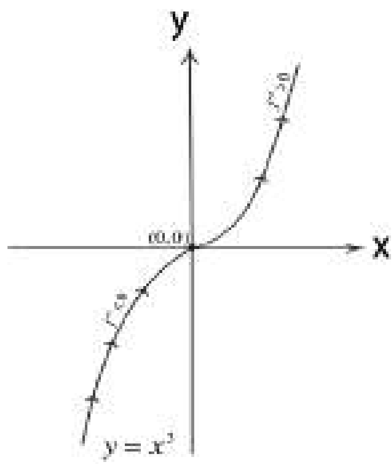
$f'(x) = 3x^2$

$f''(x) = 6x$

$f''(x) = 0 \Rightarrow 6x = 0$

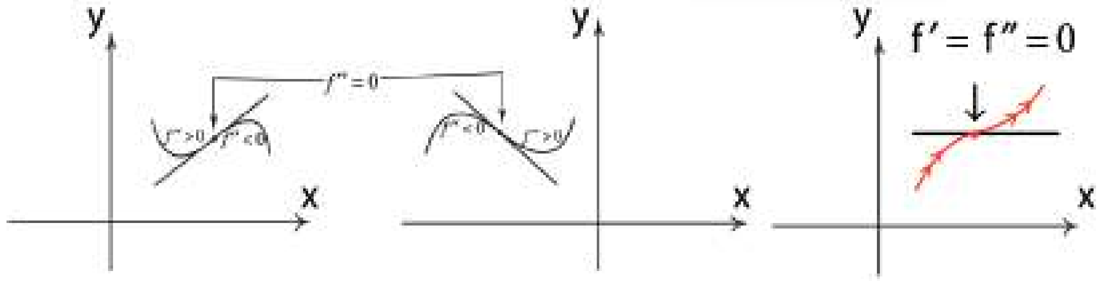
$x = 0$

$f(0) = 0$



Definition 3-5-3

The point where f changes from concave up to concave down, or (concare down to corcove up) its called inflection point .



Example 2

Examine $f(x) = 2x^3 - 3x^2 - 12x + 1$ for concavity and find inflection point.

Solution

$f(x) = 2x^3 - 3x^2 - 12x + 1$

$f'(x) = 6x^2 - 6x - 12$

$f''(x) = 12x - 6$

$f''(x) = 0 \Rightarrow 12x - 6 = 0 \Rightarrow x = \frac{1}{2}, \quad f\left(\frac{1}{2}\right) = -\frac{11}{2}$

$x = \frac{1}{2}$ is a passible inflection point.

f is concave up on $\left\{x : x > \frac{1}{2}\right\}$

f is concave down on $\left\{x : x < \frac{1}{2}\right\}$

Then the point $\left(\frac{1}{2}, -\frac{11}{2}\right)$ is inflection point.

Example 3

Find the intervals of concavity and inflection points of following functions.

a) $f(x) = 4x^3 - x^4$

b) $f(x) = x + \frac{1}{x}, \ x \neq 0$

c) $h(x) = 4 - (x + 2)^4$

d) $f(x) = 3 - 2x - x^2$

e) $f(x) = x^4 + 3x^2 - 3$

Solution

a) $f(x) = 4x^3 - x^4$

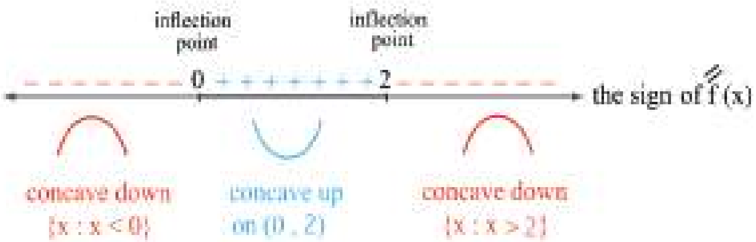
$f'(x) = 12x^2 - 4x^3$

$f''(x) = 24x - 12x^2$

$f''(x) = 0 \Rightarrow 24x - 12x^2 = 0 \Rightarrow \ 12x(2 - x) = 0 \ \Rightarrow \ x = 0, x = 2$

$f(0) = 0 \qquad f(2) = 16$

$(0, 0) \qquad (2, 16)$



(0, 0) and (2, 16) are inflection points.

b) $f(x) = x + \frac{1}{x}, \quad x \neq 0$

$f'(x) = 1 - \frac{1}{x^2}$

$f''(x) = \frac{2}{x^3}$

$f''(0)$ is undefined

The inflection point does not exist since $0 \notin$ domain of the function.

c) $h(x) = 4 - (x + 2)^4$

$h'(x) = -4(x + 2)^3$

$h''(x) = -12(x + 2)^2$

$h''(x) = 0 \Rightarrow -12(x + 2)^2 \Rightarrow x = -2, h(-2) = 4$

$(-2, 4)$

Since the concavity does not change around $x = -2$ then the inflection point there does not exist.

d) $f(x) = 3 - 2x - x^2$

$f'(x) = -2 - 2x$

$f''(x) = -2 < 0$

f is concave down on IR

Then the inflection point does not exist

e) $f(x) = x^4 + 3x^2 - 3$

$f'(x) = 4x^3 + 6x$

$\Rightarrow f''(x) = 12x^2 + 6 > 0 \qquad x \in IR$

f is concave up on IR

Then the inflection point dose not exist

3 - 6 The second derivative test for local maximum and local minimum

In our previous lessons we have seen that the first derivative test tells us whether critical point is a local extreme or not.

Now we instead of firsts dervative test we can sometimes use the second derivative test to determine the kind of local extreme.

Let c be a critical point of f then;

1) if $f'(c) = 0$ and $f''(c) < 0$ then f has a local maximum at $x = c$ so, $(c, f(c))$ is local maximum point.

2) if $f'(c) = 0$ and $f''(c) > 0$ then f has a local mimimum at $x = c$ so, $(c, f(c))$ is local minimum point.

3) if $f''(c) = 0$ or $f''(c)$ is undefined then this test fails. (we use the first derivative test instead.)

Example 1

By using second derivative test find the local extreme of following functions.

a) $f(x) = 6x - 3x^2 - 1$

b) $f(x) = x - \frac{4}{x^2}, \ x \neq 0$

c) $f(x) = x^3 - 3x^2 - 9x$

d) $f(x) = 4 - (x + 1)^4$

Solution

a) $f(x) = 6x - 3x^2 - 1$

$f'(x) = 6 - 6x$

$f'(x) = 0$

$0 = 6 - 6x \Rightarrow x = 1$

$f''(1) = -6 < 0$

$f'(1) = 0$ and $f''(1) < 0$ then f has a local maximum at $x = 1$

$(1, f(1)) = (1, 2)$ is local maximum point .

b) $f(x) = x - \dfrac{4}{x^2}$ $\qquad x \neq 0$

$f'(x) = 1 + \dfrac{8}{x^3}$

$f'(x) = 0$

$0 = 1 + \dfrac{8}{x^3} \Rightarrow x^3 + 8 = 0 \Rightarrow x = -2$

$f'(x) = 1 + \dfrac{8}{x^3}$

$f''(x) = \dfrac{-24}{x^4}$ second derivative

$f''(-2) = -\dfrac{24}{16} < 0$

$f'(-2) = 0$ and $f''(-2) < 0$ then there exists a local maxium at point $x = -2$

$(-2, f(-2)) = (-2, -3)$ is local maximum point .

c) $f(x) = x^3 - 3x^2 - 9x$

$f'(x) = 3x^2 - 6x - 9$

$f'(x) = 0$

$0 = 3(x^2 - 2x - 3) \Leftrightarrow 0 = 3(x - 3)(x + 1)$

$\Rightarrow x = 3 \qquad x = -1$

$f''(x) = 6x - 6$

$x = 3 \Rightarrow f''(3) = 18 - 6 = 12 > 0$

$f(3) = 27 - 27 - 27 = -27$

$(3, -27)$ is a local minimum point

$x = -1 \Rightarrow f'(-1) = -6 - 6 = -12 < 0$

$f(-1) = 5$

$(-1, 5)$ is a local maximum point

d) $f(x) = 4 - (x+1)^4$

$f'(x) = -4(x+1)^3$

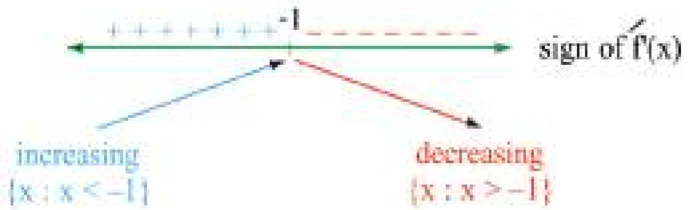
$f'(x) = 0$

$0 = -4(x+1)^3 \Rightarrow x = -1$

$f''(x) = -12(x+1)^2$

$f''(-1) = 0$

This test fails , we use the first derivative test instead



$f(-1) = 4 - (-1+1)^4 = 4$, $(-1, 4)$ is local maximum point

Example 2

Let $f(x) = x^2 + \dfrac{a}{x}$, $x \neq 0$ $a \in R$

Find the value of (a) knowing that f has an inflection point at x = 1, then prove that f has not local maximum point.

Solution

$f(x) = x^2 + \dfrac{a}{x} \Rightarrow f'(x) = 2x - \dfrac{a}{x^2} \Rightarrow f''(x) = 2 + \dfrac{2a}{x^3}$

$f''(1) = 2 + \dfrac{2a}{(1)^3} = 0$

$\Rightarrow 2 + 2a = 0 \Rightarrow a = -1$

$\because f(x) = x^2 - \dfrac{1}{x}$

$\Rightarrow f'(x) = 2x + \dfrac{1}{x^2}$

$\Rightarrow f'(x) = 0 \Rightarrow 2x + \frac{1}{x^2} = 0$

$2x^3 = -1 \Rightarrow x^3 = \frac{-1}{2}$

$\Rightarrow x = -\sqrt[3]{\frac{1}{2}}$

$f''(x) = 2 - \frac{2}{x^3} \Rightarrow f''(x) = 2 - \frac{2}{-\frac{1}{2}} \Rightarrow f''(x) = 6 > 0, \forall x \in \mathbb{R}$

$\therefore$ at $x = -\sqrt[3]{\frac{1}{2}}$, f has local minimum, f has not local maximum.

Example 3

Find the values of a and b so that the curve $y = x^3 + ax^2 + bx$ has local maximum at $x = -1$ and local minimum at $x = 2$ then find the inflection point if it exists.

Solution

$y = x^3 + ax^2 + bx$

$\Rightarrow \frac{dy}{dx} = 3x^2 + 2ax + b$

since f has local maximum at $x = -1$ then

$\therefore \frac{dy}{dx} = 0$

$0 = 3(-1)^2 + 2a \cdot (-1) + b \Rightarrow 3 - 2a + b = 0$ ....(1)

since f has local miminun at $x = 2$ then

$\therefore \frac{dy}{dx} = 0$

$0 = 3(2)^2 + 2a \cdot (2) + b \Rightarrow 12 + 4a + b = 0$ ....(2)

by solving (1) and (2) together

$a = -\frac{3}{2}, \quad b = -6$

$\therefore y = x^3 - \frac{3}{2}x^2 - 6x$

$\Rightarrow \frac{dy}{dx} = 3x^2 - 3x - 6 \quad \Rightarrow \frac{d^2y}{dx^2} = 6x - 3 \,.\, \frac{d^2y}{dx^2} = 0 \;\Rightarrow 6x - 3 = 0 \quad \Rightarrow \; x = \frac{1}{2}$

concave down $\{x : x < \frac{1}{2}\}$ concave up $\{x : x > \frac{1}{2}\}$

$f\left(\frac{1}{2}\right) = \frac{-26}{8} = \frac{-13}{4}$ $\left(\frac{1}{2}, -\frac{13}{4}\right)$ is inflection point.

Example 4

If the function of curve is $f(x) = ax^3 + bx^2 + c$ is concave up on $\{x : x < 1\}$ and concave down $\{x : x > 1\}$ and it is tangent to the line $y + 9x = 28$ at point $(3, 1)$ then find the values of real numbers a, b and c.

Solution

Since f is continuous because it is polynomial function and it is concave up on $\{x : x < 1\}$ and concave down on $\{x : x > 1\}$ then it has an inflection point on $x = 1$.

$\therefore f'(x) = 3ax^2 + 2bx$

$f''(x) = 6ax + 2b$

$f''(1) = 0 \Rightarrow 6a + 2b = 0$ .... dividing by 2

$3a + b = 0 \Rightarrow b = -3a$ ....(1)

$\frac{dy}{dx} + 9\frac{dx}{dx} = \frac{d(28)}{dx}$........... by differentiating both sides of $y + 9x = 28$

$\frac{dy}{dx} = -9$

$f'(3)$ is slope at $x = 3$.

$f'(3) = 27a + 6b$

$-9 = 27a + 6b$

$-3 = 9a + 2b$ .... (2) dividing by 3

the point (3, 1) satisfies the equation $y = ax^3 + bx^2 + c$.

$\therefore\ 27a + 9b + c = 1$ ....(3)

by solving (1) and (2)

$-3 = 9a + 2(-3a) \Rightarrow a = -1 \Rightarrow b = -3(-1) = 3$

from (3) $\Rightarrow 27 . (-1) + 9 . (3) + c = 1 \Rightarrow c = 1$

Example 5

If $f(x) = ax^3 + 3x^2 + c$ has a local maximum equal to 8, and inflection point at $x = 1$ then find $a, c \in R$

Solution

At $x = 1$ there exists inflection point.

$f''(1) = 0$

$f'(x) = 3ax^2 + 6x$

$f''(x) = 6ax + 6$ since $f''(1) = 0$

$\therefore\ 0 = 6a + 6 \Rightarrow a = -1$

$f(x) = -x^3 + 3x^2 + c$

$f'(x) = -3x^2 + 6x$

$f'(x) = 0 \Rightarrow$

$-3x^2 + 6x = 0 \Rightarrow$

$-3x\ (x - 2) = 0\ \Rightarrow x = 0,\ x = 2$ critical points

$\therefore$ f has local maximum at $x = 2$

$\therefore$ (2, 8) is local maximum point and satisfies the equation of curve.

$f(x) = -x^3 + 3x^2 + c$

$\therefore\ 8 = -8 + 12 + c$

$\Rightarrow c = 4$

Exercises

1) Let $f(x)=ax^2-6x+b$ whereby $a\in\{-4,8\}, b\in R$ find (a) value if:

a) The function f is concave down.

b) The Function f is concave up.

2) If (2,6) is a critical point for function curve $f(x)=a-(x-b)^4$ then find $a,b\in R$ and show type of critical point.

3) If $g(x)=1-12x,\ f(x)=ax^3+bx^2+cx$ are touching intersect each other at inflection point and the function f has inflection point (1,-11) , then find the values of constants $a,b,c\in R$.

4) If 6 represents a local minimum of function curve $f(x)=3x^2-x^3+c$ then find $c\in R$ and find equation of tangent at inflection point.

5) If $f(x)=ax^3+bx^2+cx$ and f is concave up at $\forall x>1$ and concave down at $\forall x<1$, and the function has local maximum point at (-1 ,5) , then find values of constants $a,b,c\in R$.

6) Let $f(x)=x^2-\frac{a}{x}$, $a\in R,\ x\neq 0$, prove that the function has no local maximum point.

7) The line 3x-y=7 is tangent to the curve $y=ax^2+bx+c$ at (2,-1) and it has a local end at $x=\frac{1}{2}$,then find value of $a,b,c\in R$,what type of the local end point ?

3 - 7 Graphing function

In order to sketch the graph of a function use the following steps.

1. **Find the domain of f** (the set o f all possible values of x.)

2. **Symmetric Test :**

If $f(-x) = f(x)$ then f is called even function and it is symmetric with respect to y – axis.

If $f(-x) = -f(x)$ then f is called odd function and it is symmetric with respect to origin.

3. **Find x and y – intercepts.**

for $x = 0$ we find y – intercept.

for $y = 0$ we find x – intercept.

4. **Asymptotes**

When plotting the graph of a function, we need to know the behavior of the function at infinity and the behavior near points where the function is not defined. To describe these situations we define the term "asymptote."

If $y = \frac{g(x)}{h(x)}$, put $h(x) = 0$ $\quad x = a$ (vertical Asymptote)

if $x = \frac{n(y)}{m(y)}$, put $m(y) = 0$ $\quad y = b$ (Horizontel Asymptote)

5. Find $f'(x)$, $f''(x)$ and intervals of increasing, and decreasing local maximum, minimum and inflection point.

6. Find additional points (which we use them to sketch the graph.)

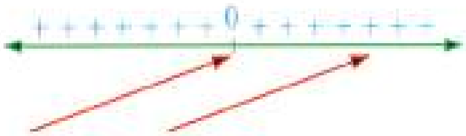
Example 1

Sketch the graph the function $f(x) = x^5$ using differentiation.

Solution

1. Since f is polynomial function the Domain = IR.
2. The point (0, 0) x– and y – intercepts.
3. $\forall x \in IR,\ \exists\ (-x) \in IR \Rightarrow f(-x) = (-x)^5 = -x^5 \Rightarrow f(-x) = -f(x)$ f is odd function.

   It is symmetric with respect to origin
4. f has no asymptotes.Because the function is not rational function
5. $f'(x) = 5x^4$ $f'(x) = 0 \Rightarrow x = 0 \Rightarrow (0, 0)$



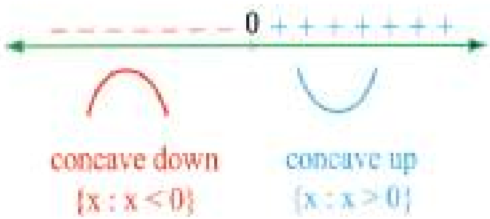
sign of $f'(x)$

$\{x : x < 0\}$, $\{x : x > 0\}$ increasing f.

the point (0, 0) is critical point.

$f''(x) = 20x^3$

$f''(x) = 0 \Rightarrow x = 0$

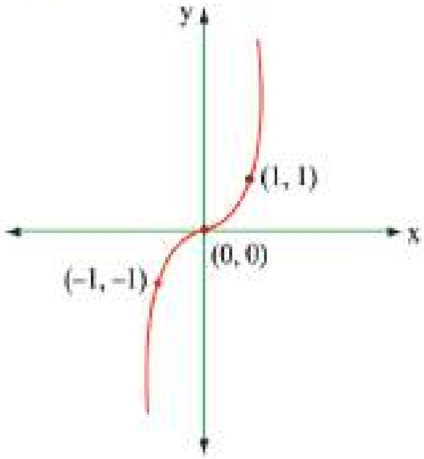


sign of $f''(x)$

the point (0, 0) is inflection point.

6.

x	0	1	–1	2	–2
y	0	1	–1	32	–32



Example 2

Sketch the graph of the function $y = x^3 - 3x^2 + 4$ using differentiation.

Solution

1) Since y is polynomial function, domain of y is IR.

2) $x = 0 \Rightarrow y = 0^3 - 3(0^2) + 4 \Rightarrow y = 4$ (0, 4) is y – intercept.

3) $\forall x \in IR, \ \exists (-x) \in IR \Rightarrow f(-x) = (-x)^3 - 3(-x)^2 + 4$

$$= -x^3 - 3x^2 + 4 \neq f(x)$$

so $f(-x) \neq -f(x)$, $f(x) \neq f(-x)$ the function is neither even nor odd, and f is not symmetric.

4) f has no asymptotes Because f(x) is not rational function.

5) $f(x) = x^3 - 3x^2 + 4 \Rightarrow f'(x) = 3x^2 - 6x$

$f'(x) = 0 \Rightarrow 3x^2 - 6x = 0 \Rightarrow 3x(x-2)=0$

$\Rightarrow x = 0, \ x = 2$

$f(0) = 4 \Rightarrow (0, 4)$

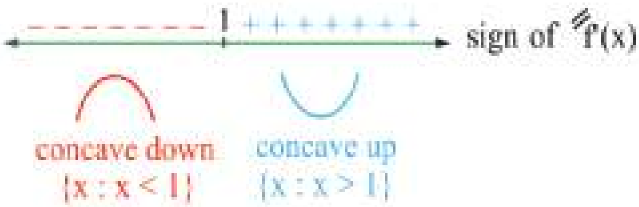
$f(2) = 0 \Rightarrow (2, 0)$

f is increasing on $\{x : x < 0\}$, $\{x : x > 2\}$

f is decreasing on open interval (0, 2)

the point (0, 4) is local maximum, (2 ,0) is local minimum point

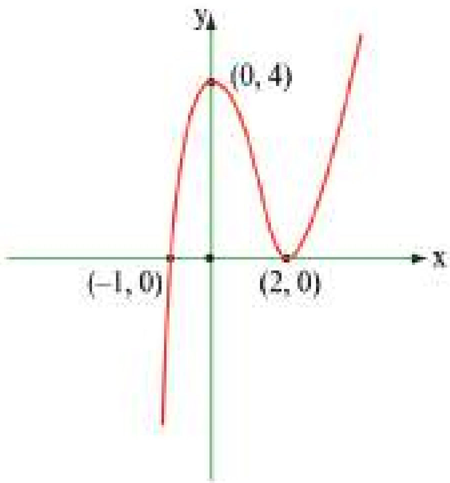
$f''(x) = 6x - 6 \Rightarrow f''(x) = 0 \Rightarrow 0 = 6x - 6 \Rightarrow x = 1$



$f(1) = 2 \Rightarrow (1, 2)$ is inflection point.

6)

x	0	1	–1	3	2
y	4	2	0	4	0



Example 3

Sketch the graph of the function $f(x) = \dfrac{3x-1}{x+1}$ by using differentiation.

Solution

1) denominator can not be 0. Let: $x + 1 = 0 \Rightarrow x = -1$

Domain $= IR - \{-1\}$

2) Since 1 belongs to domain of f but (–1) doesn't belong to domain of f. then the curve is not symmetric to y–axis and not symetric to origin

3. $x = 0 \Rightarrow y = -1 \Rightarrow (0, -1) \Rightarrow$ y – intercept.

$y = 0 \Rightarrow x = \frac{1}{3} \Rightarrow \left(\frac{1}{3}, 0\right) \Rightarrow$ x – intercept.

4. $x + 1 = 0 \Rightarrow x = -1$ (vertical asymptote)

$$f(x) = y = \frac{3x-1}{x+1} \Rightarrow yx + y = 3x - 1 \Rightarrow yx - 3x = -1 - y \Rightarrow x(y-3) = -1 - y$$

$$\Rightarrow x = \frac{-1-y}{y-3}$$

$y - 3 = 0 \Rightarrow y = 3$ (horizontal asymptote)

Short way: **(Not always correct)**

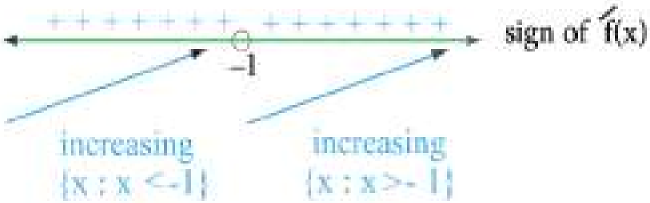
$f'(x) = \frac{3x-1}{1x+1}$, $\frac{3}{1} = 3$ horizontal asymptote

5. $$f'(x) = \frac{(x+1).(3) - (3x-1).(1)}{(x+1)^2} = \frac{3x+3-3x+1}{(x+1)^2} = \frac{4}{(x+1)^2}$$

$\forall x \in IR - \{-1\}, f'(x) > 0$

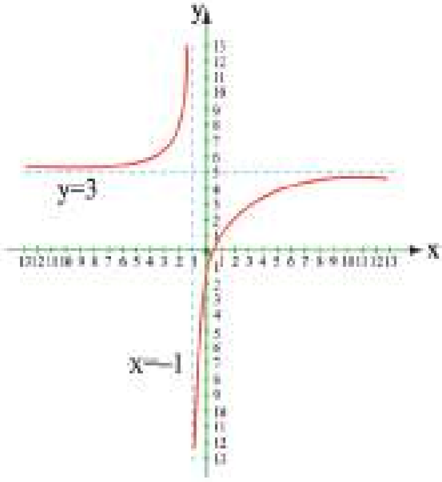
$\{x : x < -1\}$ increasing f

$\{x : x > -1\}$ increasing f



$$f'(x) = 4(x+1)^{-2} \Rightarrow f''(x) = -8(x+1)^{-3}(1) = \frac{-8}{(x+1)^3}$$

function has no inflection point because (−1) doesn't belong to domain of f.



Example 4

Sketch the graph of the function $f(x) = \frac{x^2}{x^2+1}$ by using differentiation.

Solution

1) Domain of f = IR

2) $x = 0 \Rightarrow y = 0 \quad \therefore (0, 0)$ is x − y − intercepts.

$\forall x \in IR, \ \exists (-x) \in IR$

3) $f(-x) = \frac{(-x)^2}{(-x)^2+1} = \frac{x^2}{x^2+1} = f(x)$

$\therefore$ f is even function $\Rightarrow$ f is symmetric with respects to y − axis.

4) $x^2 + 1 \neq 0$. then f has not vertical asymptote

$$f(x) = y = \frac{x^2}{x^2+1} \Rightarrow yx^2 + y = x^2$$

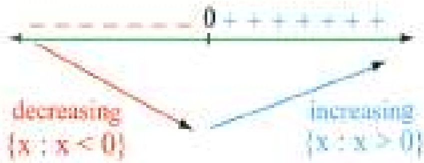
$$\Rightarrow x^2(y-1) = -y$$

$$\Rightarrow x^2 = \frac{-y}{y-1} \Rightarrow y - 1 = 0 \Rightarrow y = 1$$

then f has (horizontal asymptote)

5) $f'(x) = \dfrac{(x^2+1)(2x) - x^2(2x)}{(x^2+1)^2}$

$= \dfrac{2x}{(x^2+1)^2} = 0 \Rightarrow x = 0 \;,\; y = 0 \Rightarrow (0, 0)$



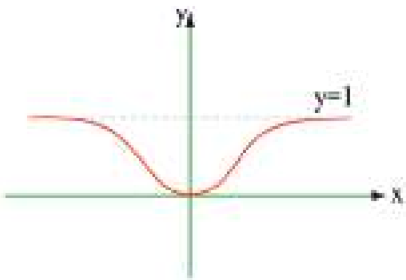
sign of $f'(x)$

(0, 0) is local minimum point.

$f''(x) = \dfrac{(x^2+1)^2\,(2) - 2x(2)(x^2+1)2x}{(x^2+1)^4} = \dfrac{2-6x^2}{(x^2+1)^3} = 0 \Rightarrow x = \pm\dfrac{1}{\sqrt{3}}$ sign of $f''(x)$

f(x) concave down on $\left\{x: x < -\dfrac{1}{\sqrt{3}}\right\}, \left\{x: x > \dfrac{1}{\sqrt{3}}\right\}$

f(x) concave up on open interval $\left(-\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$



$f\left(\pm\dfrac{1}{\sqrt{3}}\right) = \dfrac{1}{4} \Rightarrow \left(\dfrac{1}{\sqrt{3}}, \dfrac{1}{4}\right), \left(-\dfrac{1}{\sqrt{3}}, \dfrac{1}{4}\right)$ are inflection points.

Exercises

By using the differentiation sketch the graph of the following functions(curves)

1) $f(x)=10-3x-x^2$

2) $f(x)=x^2+4x+3$

3) $f(x)=(1-x)^3+1$

4) $f(x)=6x-x^3$

5) $f(x)=\frac{1}{x}$

6) $f(x)=\frac{x-1}{x+1}$

7) $f(x)=(x+2)(x-1)^2$

8) $f(x)=\frac{x^2-1}{x^2+1}$

9) $f(x)=2x^2-x^4$

10) $f(x)=\frac{6}{x^2+3}$

3 - 8 Optimization (Maximum, Minimum) Problems

By using differential calculus we can solve many problems that call for **minimization** or **maximization** a problems to get maximum area, minimum velocity, ... etc.

strategy for : solving Max- Min problems:

1) Draw a picture and name the variables and constants.

2) Write an equation that relates the variables.

3) Differentiate with respect to the variable to test the critical points and end points.

Example 1

Find the number when it is added to its square the result is the as small as possible.

Solution

Let the number be x, and its square be x^2. $f(x) = x + x^2$

$f'(x) = 1 + 2x.$ $f''(x) = 2 > 0$

$f'(x) = 0$ $1 + 2x = 0 \Rightarrow x = -\frac{1}{2}$

$f''\left(-\frac{1}{2}\right) = 2 > 0$

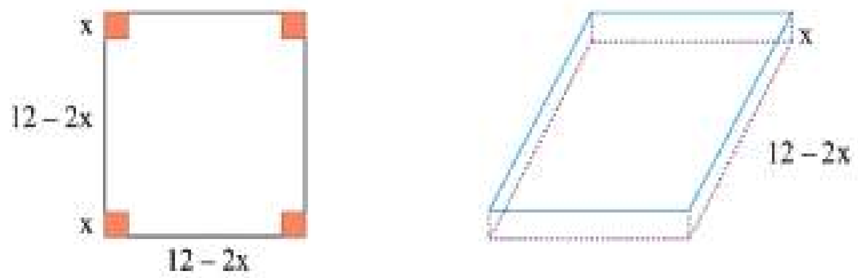
the local minimum at $x = -\frac{1}{2}$

$\therefore$ the number is $-\frac{1}{2}$

Example 2

A squcre sheet of copper 12 cm on a side is to be used to make an open top box by cutting a small square of copper from each corner and bending (turning) up the sides. What is the greatest volume of this box ?

Solution



Assume that the length of the side of square cut is x cm.

After it is bent up (turned) the dimensions of box are

$$x,\ 12 - 2x,\ 12 - 2x$$

Volume of box = the product of three dimensions.

$v = (12 - 2x)\ (12 - 2x)\ x$

$v = f(x) = x(144 - 48x + 4x^2)$

$v = f(x) = 144x - 48x^2 + 4x^3$

$\frac{dv}{dx} = f'(x) = 144 - 96x + 12x^2 \Rightarrow \quad \frac{dv}{dx} = 0$

$\Rightarrow \quad 0 = 12\,(12 - 8x + x^2) \Rightarrow 12\,(6 - x)\,(2 - x) = 0$

$\Rightarrow \quad x = 6,\ x = 2$ are critical numbers.

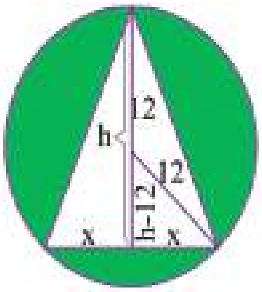
Note that 6 is neglected because it is impossible to be $x = 6$ the volume is maximum at $x = 2$ and the volume is

$v = f(2) = 2\,(12 - 4)^2 = 128\ cm^3$

Example 3

Find the dimensions of the largest isosceles triangle which is inscribed in a circle of radius 12 cm. Then prove that the ratio of area of triangle to the area of circle is $\frac{3\sqrt{3}}{4\pi}$.

Solution



Let's assume that height of triangle is h. the length of its base is

b =2x to get the relation between the variables by pythagorean theorem.

$x^2 + (h - 12)^2 = (12)^2$

$x^2 + h^2 - 24h + 144 = 144$

$x^2 = 24h - h^2$

$x = \sqrt{24h - h^2}$

$A = \frac{1}{2}(b).h = \frac{1}{2}(2x).h = hx \quad \Rightarrow \quad A = f(h) = h\sqrt{24h - h^2} = \sqrt{h^2(24h - h^2)}$

$\Rightarrow \quad A = f(h) = \sqrt{24h^3 - h^4}$

$\frac{dA}{dh} = f'(h) = \frac{72h^2 - 4h^3}{2\sqrt{24h^3 - h^4}}$ ....(by differentiating both sides.)

$f'(h) = 0 \quad \Rightarrow 72\,h^2 - 4h^3 = 0 \quad \Rightarrow 4\,h^2\,(18 - h) = 0 \quad \Rightarrow h = 18$ cm

$\therefore$ height of triangle is h = 18 cm

$x = \sqrt{24h - h^2} \Rightarrow x = \sqrt{24.\,(18) - 18^2} \Rightarrow x = 6\sqrt{3}$ cm

$\therefore$ the base of triangle is $b = 2x = (2)(6\sqrt{3}) = 12\sqrt{3}$ cm

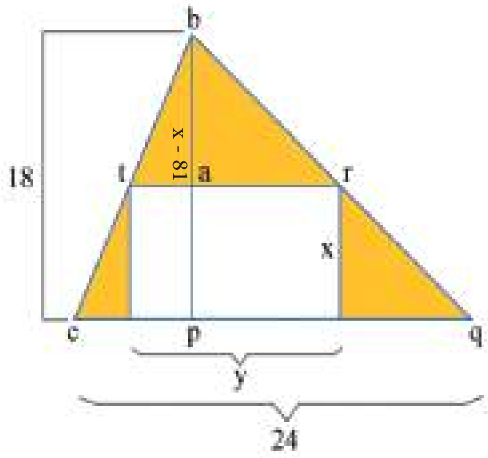
the area of circle $\Rightarrow A_1 = \pi r^2 = \pi(12)^2 = 144\pi\ cm^2$

the area of triangle $\Rightarrow A_2 = \frac{1}{2}\,bh = \frac{1}{2}\,(2x)\,h = \frac{1}{2}.(2)(6\sqrt{3})(18) = 108\sqrt{3}\,cm^2$

$\frac{\text{area of triangle}}{\text{area of circle}} = \frac{A_2}{A_1} = \frac{108\sqrt{3}}{144\pi} = \frac{3\sqrt{3}}{4\pi}$ (Q.E.D)

Example 4

Find the dimensions of the largest rectangle which can be inscribed in a triangle whose length of its base is 24 cm and the height is 18 cm. such that two of the vertices are on the base and the others on the legs.



Solution

Let the dimensions of the rectangle are :

length be x.

width be y.

the relation between variables :

triangle (btr) and (bcq) are similar triangles since the correspording angles are equal. So the correspording sides their are proportional.

|ba|=18-x

$$\frac{tr}{cq} = \frac{ba}{bp} \Rightarrow \frac{y}{24} = \frac{18-x}{18} \Rightarrow y = \frac{24}{18}(18-x) \Rightarrow y = \frac{4}{3}(18-x)$$

area of rectiagle $A = xy = x \cdot \frac{4}{3}(18-x)$

$$f(x) = A = \frac{4x}{3}(18-x) = \frac{4}{3}(18x - x^2)$$

$$f'(x) = \frac{4}{3}(18-2x)$$

$$f'(x) = 0 \Rightarrow x = 9$$

$$f''(x) = -\frac{8}{3}$$

$$f''(9) = -\frac{8}{3} < 0$$

$\therefore$ f has maximum value at x = 9 cm

$$y = \frac{4}{3}(18-9) = 12 \text{ cm}$$

$\therefore$ y = 12 cm

Example 5

The sum of perimeter of a circle and a square is 60 cm. If the sum of their areas is minimum then prove that the diameter of a circle is the length of the side of square.

Solution

Let the radius of circle is r cm , the length of side of square is x cm.

Perimeter of square + circumferenc (perimeter) of circle = 60 cm.

$\therefore\ 4x + 2\pi r = 60$ cm

$\Rightarrow r = \frac{1}{\pi}(30 - 2x)$

the function = area of square + area of circle.

$A = x^2 + \pi r^2$

$A = x^2 + \pi\left[\frac{1}{\pi}(30 - 2x)\right]^2$

$A = f(x) = x^2 + \frac{1}{\pi}(900 - 120x + 4x^2)$

$A' = f'(x) = 2x + \frac{1}{\pi}(-120 + 8x)$

$A' = f'(x) = 0 \Rightarrow 0 = 2x + \frac{1}{\pi}(-120 + 8x) \Rightarrow 0 = \pi x + 4x - 60$

$\Rightarrow 60 = \pi x + 4x$

$x(\pi + 4) = 60 \Rightarrow x = \frac{60}{\pi + 4}$ cm

$\therefore\ r = \frac{1}{\pi}\left(30 - \frac{120}{\pi + 4}\right) = \frac{30}{\pi + 4}$ cm $\Rightarrow$ x = 2r = diameter

$f''(x) = 2 + \frac{1}{\pi}(8) > 0$

The function f has local minimum .

(Q.E.D)

Example 6

Find a point or points which belogn to the hyperbola $y^2 - x^2 = 3$ that are nearest closest to the point. (0,4)

Solution

Let's assume that the point P(x, y) belongs to the hyperbola $y^2 - x^2 = 3$ and satisfies the equation.

$\therefore\ x^2 = y^2 - 3$ .... (1)

$S = \sqrt{(x+0)^2 + (y-4)^2}$ .... distance between P(x, y) and (0, 4)

$\therefore\ S = \sqrt{x^2 + y^2 - 8y + 16}$ .... (2)

$S = f(y) = \sqrt{y^2 - 3 + y^2 - 8y + 16}$ .... substituting (1) in (2)

$= \sqrt{2y^2 - 8y + 13}$

$f'(y) = \dfrac{4y - 8}{2\sqrt{2y^2 - 8y + 13}}$

$f'(y) = 0 \Rightarrow 4y - 8 = 0 \Rightarrow y = 2$

$\because\ x^2 = y^2 - 3$

$x^2 = 2^2 - 3$

$x^2 = 4 - 3$

$x^2 = 1$

$x = \pm 1$ $\therefore$ the points (1, 2) and (–1, 2) are the nearest closest possible to the point (0,4) .

Exercises

1) Find two positive numbers whose sum is 75 and the product of one of them square of second one is largest possible

2) Find the height of largest right circular cylinder which can be inscribed in a sphere whose radius is $4\sqrt{3}$ cm.

3) Find the dimensions of largest rectangle which is inscribed into a semicircle whose radius is $4\sqrt{2}$ cm.

4) Find the largest possible area of an isosceles triangle whose one side is $8\sqrt{2}$ cm.

5) Find the largest possible perimeter of rectangle whose area is 16 cm^2.

6) Find the lest possible volume of right circular cone that can be inscribed in a sphere whose radius is 3 cm.

7) Find the equation of a line which passes through the point (6,8) and makes a smallest triangle with two coordinate axes in first quadrant.

8) Find the dimensions of largest possible rectangle which has its base on the x-axis and its upper two vertices on the parabola $f(x)=12-x^2$. Then find its perimeter

9) Find the dimensions of largest right circular cylinder which is inscribed in a right circular cone whose height is 8 cm,and the length of diameter of its base is 12 cm.

10) Find the largest possible volume of right circular cone which is formed by revolving the vertex of right triangle around the one of its right sides whose hypotenuse is $6\sqrt{3}$ cm.

11) An open top cylinderical container whose volume is 125π cm^3 .Find dimensions of the container so that the area of metal used is least possible .

12) A parallelpiped tank whose length of base is twice of its width.If the area of used metal to make it is 108 m^2 ,find the dimensions of the tank so that the volume is largest possible knowing that the tank is covered compeletely.

Chapter (4) : Integration

4 - 1 Regions Bounded by Curves

4 - 2 The Lower and Upper Rectongles

4 - 3 Definition of Integration

4 - 4 The Fundomental Theorem of Integral

4 - 5 Properties of Definite Integral

4 - 6 Indefinite Integral

4 - 7 The Natural Logarithm

4 - 8 Plane Area by Definite Integral

4 - 9 Volume of Revolution

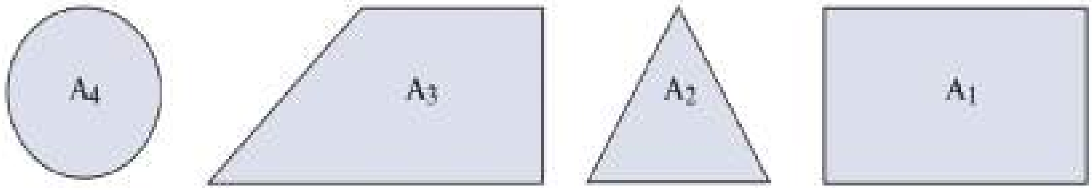
Terminology

Term	Symbol or Mathematical Relation
Partition of inter	$\sigma = (x_1 , x_2 , \dots , x_n)$
Lower Rectangles	$L(\sigma, f)$
Upper Rectangles	$U(\sigma, f)$
Sigma	σ

04 INTEGRATION

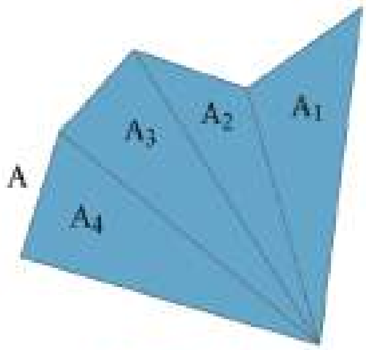
4 - 1 Regions Bounded by Curves

In previous classes, you have studied calculating the area of the regular shapes shown in the figure 4 – 1.



(Figure 4 -1)

In the figures above A_1 is rectangular region, A_2 is triangular region, A_3 is trapezoid region and A_4 is circular region and of course you know how to calculate area of these shapes.



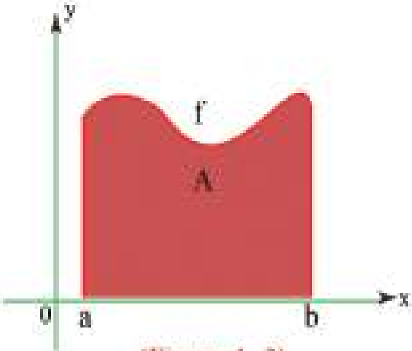
(Figure 4 -2)

But the region of A in the Figure 4 – 2 is called polygonal region and its area can be calculated by dividing it in to triangular regions A_1, A_2, A_3, A_4.

Its area is $A = A_1 + A_2 + A_3 + A_4$

By the same way we can calculate the area of regular polygons by dividing it in to region of triangle, square, rectangle... etc.

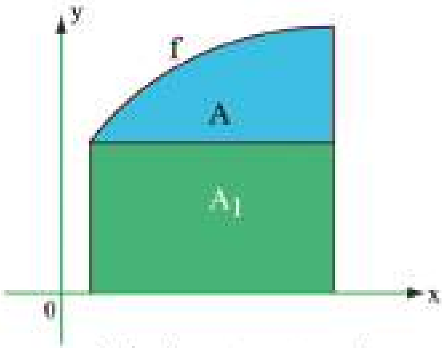
But the region of A which is shown in the figure 4 – 3 is called region under the curve f and it is the set of points bounded by the curve, x – axis and the lines x = a, x = b.



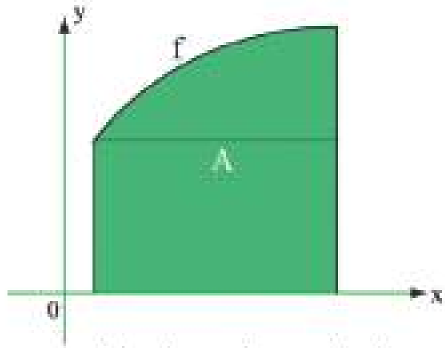
(Figure 4 -3)

In the figure it is not possible to divide it into regular shapes (triangle, square, rectangle, ...)

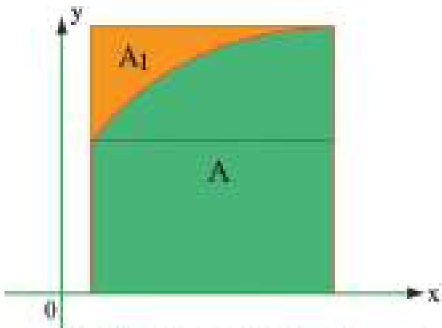
How can we calculate this area?



A_1 is largest rectangular region in region A.



A is the region under the curve.



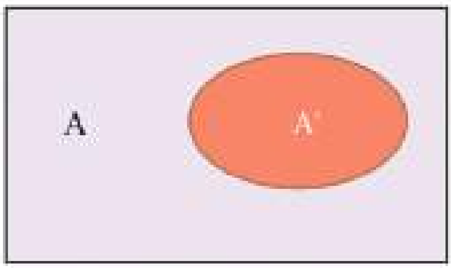
A_1 is the smallest rectangular region out side region A.

(Figure 4 -4)

Note

1) Area of region of any shape is a non-negative real number.

2) In the figure 4 - 5 if $A' \subseteq A$ then the area of region $A' \leq$ area of region A.



(Figure 4-5)

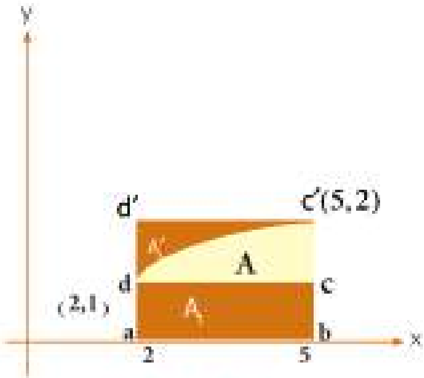
4-1-1 Finding the Approximate Area of any Region

Example 1

In the figure 4-6 , A is region under the cantinuous curve f, find the approximate value of area region such that

$A = \{(x, y) : 2 \le x \le 5, y = \sqrt{x-1}\}$

Solution



(Figure 4-6)

We draw the largest rectangle (abcd) in the region A whose base is from x = 2 to x = 5 such that $A_1 \subset A$ and this area is $A_1 = ab \,.\, ad = (5-2) \,.\, 1 = 3$ unit square.

By the same procedure, we draw the smallest rectangle (abc'd') which contains the region outside the curve and its base is from x = 2 to x = 5 let it be A_1' such that $A \subset A_1'$ let it be A_1' and this area is $A_1' = ab \,.\, ad' = (5-2) \,.\, 2 = 6$ unit square.

$\because \; A_1 \subseteq A \subseteq A_1'$

$\therefore$ Area of $A_1 \le$ area of $A \le$ area of A'_1

$3 \le$ area of region $A \le 6$

Therefore, the area region A is average of these two areas.

$A = \dfrac{A_1 + A'_1}{2} = \dfrac{3+6}{2} = 4.5$ unit square is approximate value of area of the region A.

Note

In the Example1. A_1 is area of rectangular region whose height is (ad) which is minimum value of function on [2, 5] and we will symbolize it by(m).A'_1 is area of rectangular region whose height is (ad')which

is maximum value of function on [2, 5] and we will symbolize it by(m) .

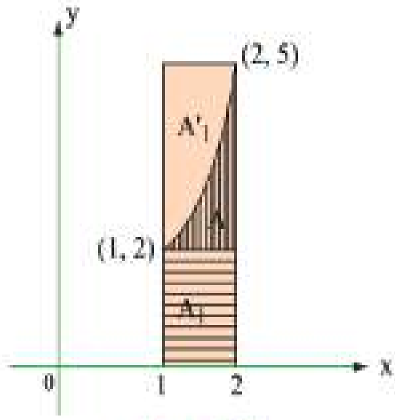
As you learned is Chapter-3 m(the minimum value of continuous function on [a, b]) and also M(the maximum value of the continuous function on [a, b]) can be found by one of the end points of [a, b] or the critical point.

Example 2

Find the approximate value of area of region A if

$A = \{(x, y) : 1 \leq x \leq 2, y = x^2 + 1\}$

Solution



(Figure 4-7)

A_1 the largest rectangular region in A its base is from

x = 1 to x = 2 and height is m = 2.

$A_1 = 2(2-1) = 2$ unit^2

A'_1 the samallest rectangular region outside A its base is from x = 1 to x = 2 and height height is M = 5

$A'_1 = 5(2-1) = 5$ unit^2

Since $A_1 \subseteq A \subseteq A'_1$

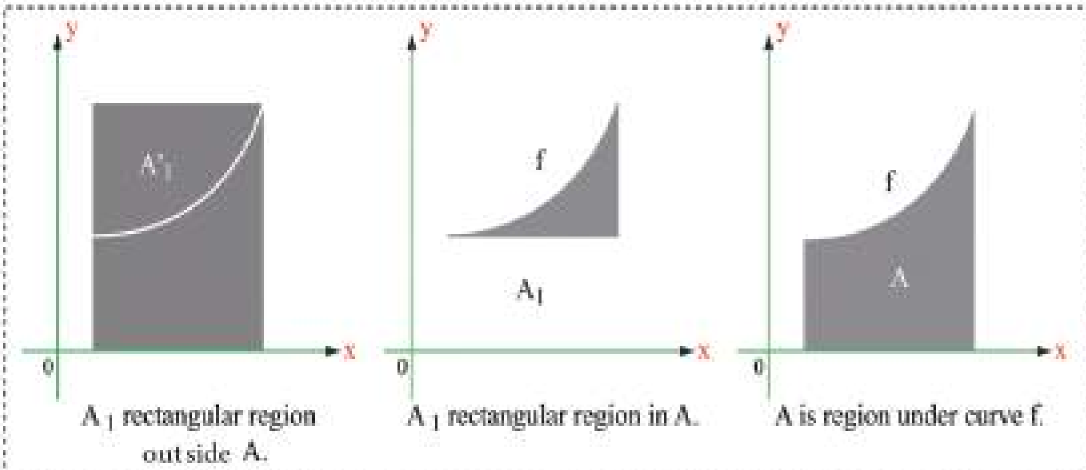
So, the area of $A_1 \leq$ area of $A \leq$ area of A'_1

$2 \leq A \leq 5$

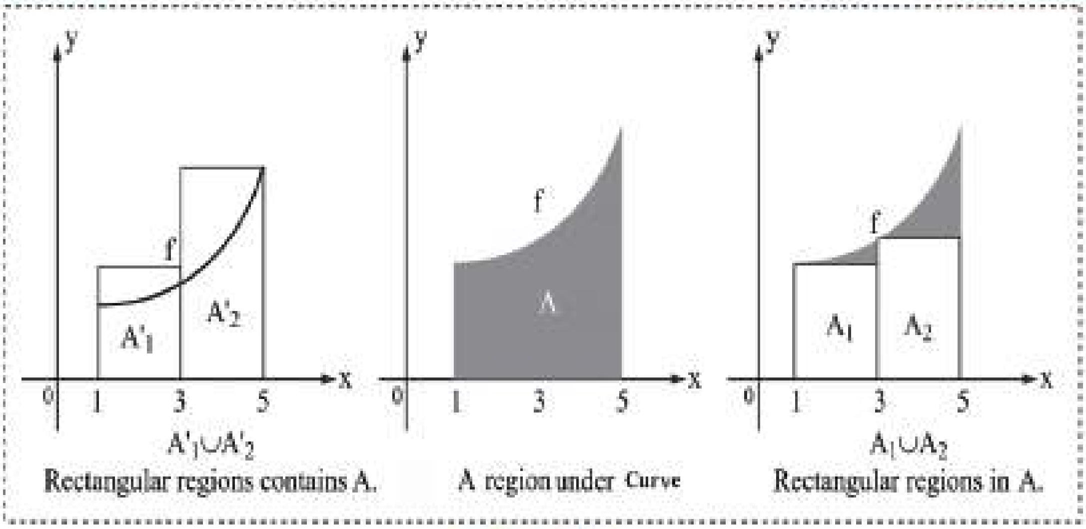
And approximate value of area of A is

$$A = \frac{A_1 + A_1'}{2} = \frac{2+5}{2} = 3.5 \text{ unit}^2$$

Lower and uppper bounded are the sum of rectangular area in (A) and the sum of rectangular axes outside (A).
The figures (4-8), (4-9), (4-10) explain that

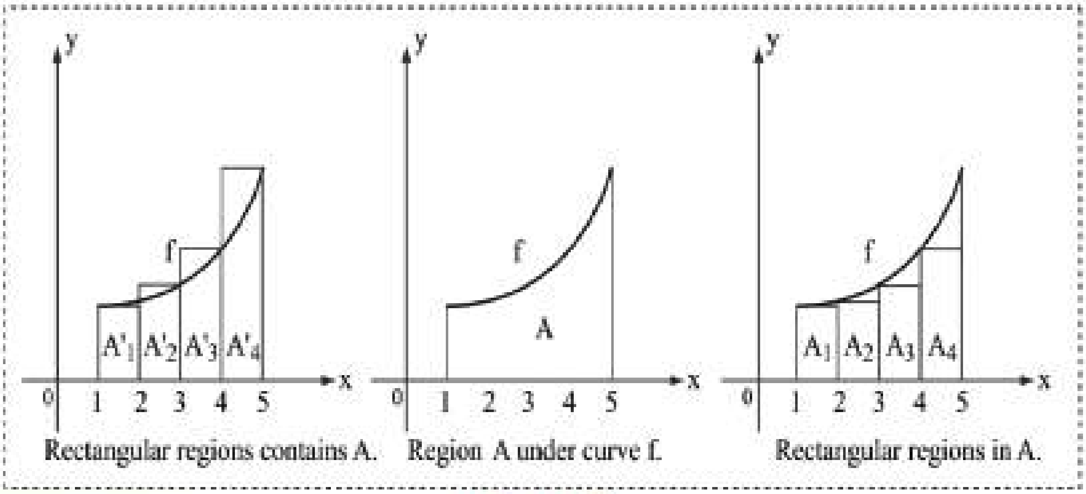


(Figure 4-8)



(Figure 4-9)

In the Figure 4-10, base [1, 5] is divided into four subintervals.



(Figure 4-10)

Note 1

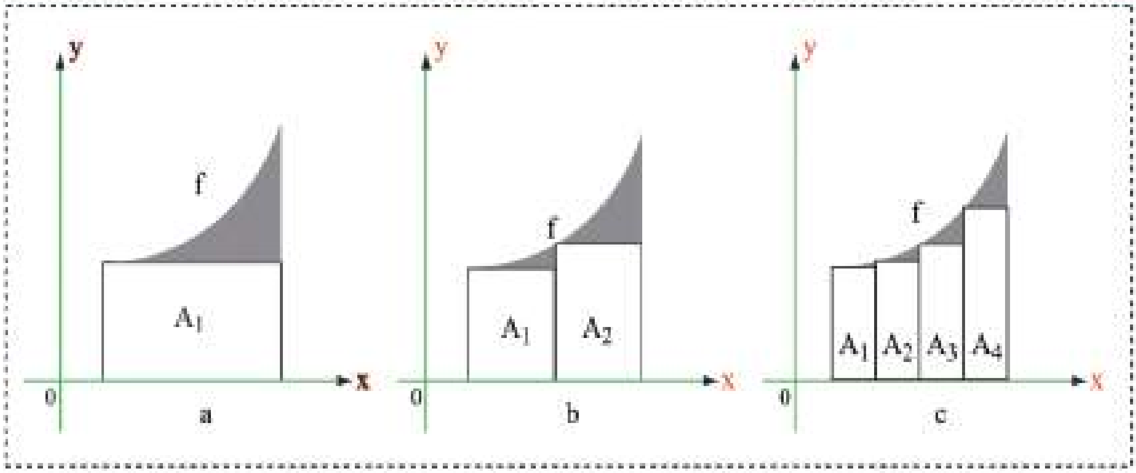
In the figure 4 - 9 the interval [1, 5] is divided into two sabintervals [1, 3], [3, 5] in this example the subintervals are called partition for the interval [1, 5]. And symbolize by $\sigma = (1, 3, 5)$. sigma notafion

In general, if we want to divide the interval [a, b] into (n) partition (equal) then the length of each interval will be $h = \frac{b-a}{n}$

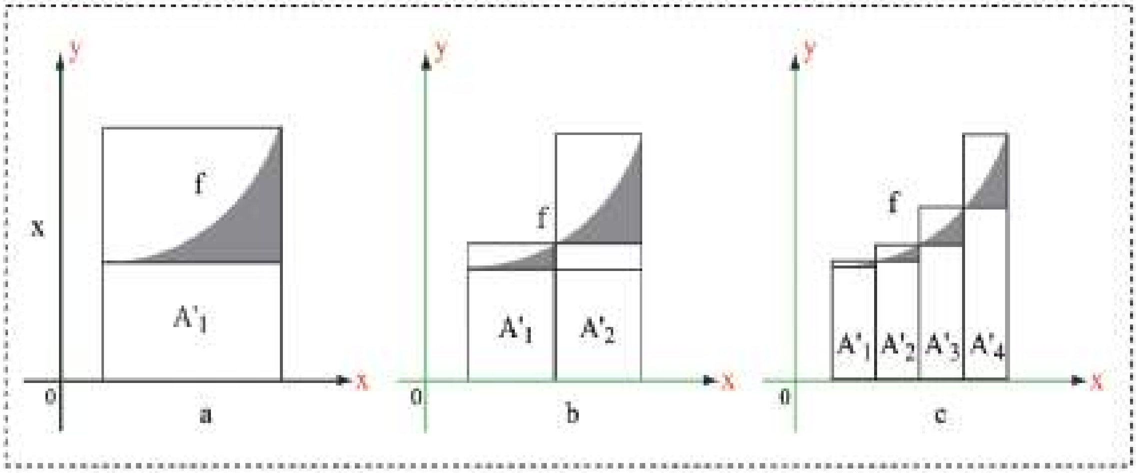
Note 2

In the figures (4 -10) , (4 -11) ,(4 -12) the interval [1 ,5] is divided into four sabintervals [1 -2] , [2 -3] , [3 -4] , [4 -5] and shown by

$\sigma = (1, 2, 3, 4, 5)$. So the approximate value of area of A is more accurate.



(Figure 4-11)



(Figure 4-12)

Example 3

Find the approximate area of following region

$A = \{(x, y) : 2 \le x \le 5,\ y = x^2 + 1\}$ using the partitions

a) $\sigma_1 = (2, 3, 5)$ b) $\sigma_2 = (2, 3, 4, 5)$

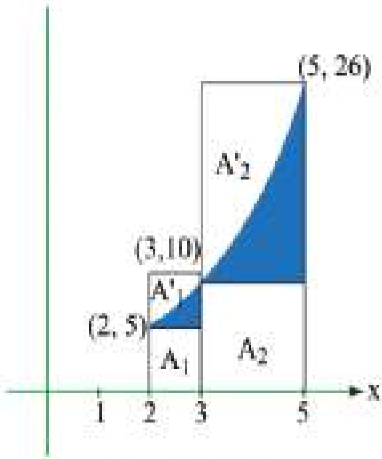
Solution

a) $\sigma_1 = (2, 3, 5)$

Since the partition is $\sigma_1 = (2, 3, 5)$ then the interval

[2, 5] will be divided into subintervals [2, 3], [3,5]

$A_1 + A_2 = 1\ (5) + 2\ (10) = 25$ unit^2



(Figure 4-13)

$A'_1 + A'_2 = 1(10) + 2(26) = 62$ unit^2

Since sum of the area of rectengular regions inside A< sum of the area of rectangular regions which outside A.

$\therefore\ 25 \le A \le 62 \Rightarrow A = \frac{25+62}{2} = 43\frac{1}{2}$ unit^2 is approximate value of area of A.

b) $\sigma_2 = (2, 3, 4, 5)$

Since the partition is $\sigma_2 = (2, 3, 4, 5)$ then the interval [2, 5] will be divided into subintervals [2, 3], [3, 4], [4, 5]

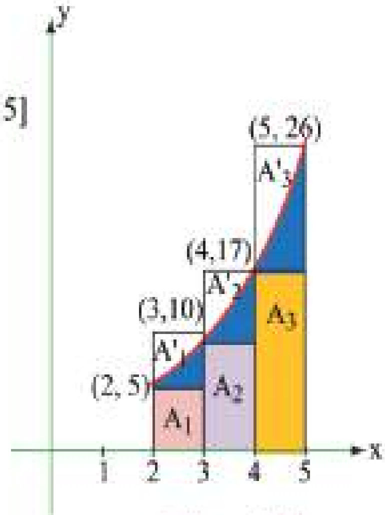
$A_1 + A_2 + A_3 = 1(5) + 1(10) + 1(17)$

$= 32$ unit^2

$A'_1 + A'_2 + A'_3 = 1(10) + 1(17) + 1(26)$

$= 53$ unit^2

$A = \frac{32+53}{2} = 42\frac{1}{2}$ unit^2



(Figure 4-14)

Note

As we see in the figures above as the number of portitions increase then the difference between the sum of area of rectangular regions in A and the sum of area of rectangular regions containing A decreases regularly.

In the previous example when partitions were (2, 3, 5) the difference was 62 – 25 = 37.

when partitions were (2, 3, 4, 5) the difference was 53 – 32 = 21.

4 - 2 The Lower and Upper Rectangles

In the previous section you learned finding the sum of area of rectangular regions in A and sum of area of rectanfgular regions which contain A (curve)

In this section we are going to introduce the function f, $f : [a, b] \to IR$ which is continious and we are going to find sum of the area of rectangular regions in A (Lower Rectangles) and the sum of the area of rectongles containing A (Upper Rectangles) such that A is region under the curve f In the figure 4 - 15 :

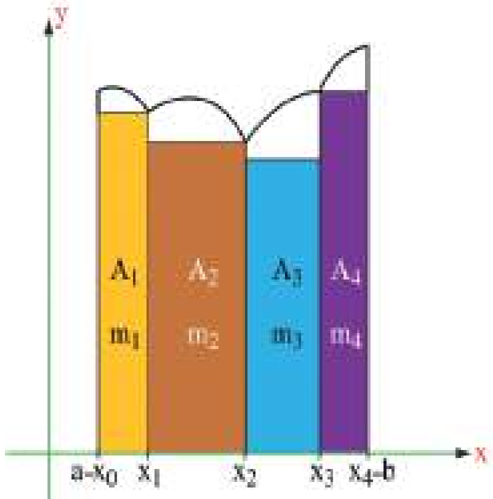
1st: Assume that $f(x) \geq 0, \forall x \in [a, b]$ such that $\sigma = (x_0, x_1, x_2, x_3, x_4)$ and the area of rectangular region A_1 whose base is bounded on $[x_0, x_1]$ and its height is (m_1) then

$A_1 = m_1 . (x_1 - x_0)$ where m_1(the minimum value of function on $[x_1, x_2]$)

By the same way $A_2 = m_2 . (x_2 - x_1)$ where its base is bounded on $[x_1, x_2]$ and height is (m_2).

Therefore, the sum of area of rectangular regions in A which is shown by $L(\sigma, f)$ is

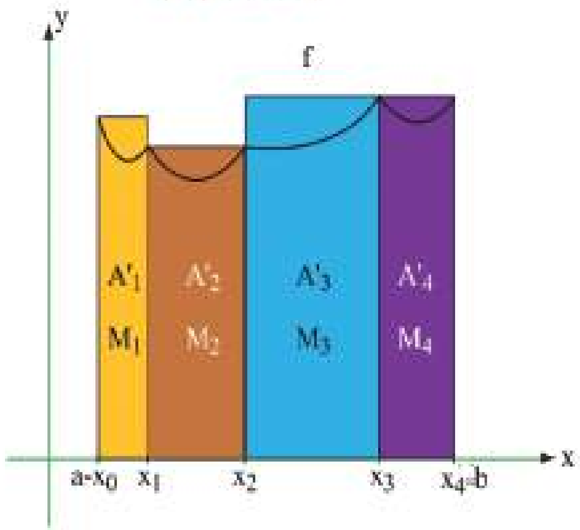
$$L(\sigma, f) = m_1 (x_1 - x_0) + m_2(x_2 - x_1) + m_3(x_3 - x_2) + m_4(x_4 - x_3)$$



(Figure 4-15)

Notice that,

$L(\sigma, f) \leq$ area of A



(Figure 4-16)

By the same way in the figure 4-16

The area of region A'_1 whose base is bounded on $[x_0, x_1]$ is $M_1 . (x_1 - x_0)$ such that M_1 is the maximum value of function on $[x_0, x_1]$.

The area of region A'_2 on $[x_1, x_2]$ is

$A'_2 = M_2 . (x_2 - x_1)$ ... etc.

The sum of area of rectangular regions containing A which is shown by $\cup(\sigma, f)$

$$\cup(\sigma, f) = M_1 . (x_1 - x_0) + M_2 . (x_2 - x_1) + M_3 . (x_3 - x_2) + M_4 . (x_4 - x_3)$$

Notice that,

$$U(\sigma, f) \geq L(\sigma, f)$$

$$L(\sigma, f) \leq \text{area of } A \leq U(\sigma, f)$$

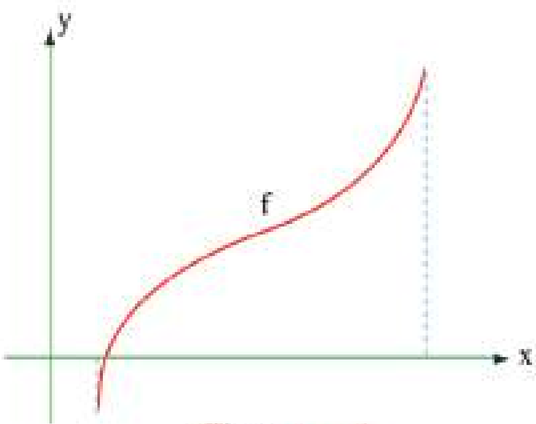
$\therefore$ the approximated area of A under partition σ is

$$A = \frac{L(\sigma, f) + \cup(\sigma, f)}{2}$$

2nd: If we don't obligate being $f(x) \geq 0, \ \forall x \in [a, b]$ as shown in the figure 4-17 then it is possible that m(the local minimum value of function) can be either negative or positive or zero.

However $L(\sigma, f)$ can be either negative or positive or zero and $\cup(\sigma, f)$ can be either negative or positive or zero.

Since the area can not be negative, we call



(Figure 4 - 17)

$L(\sigma, f)$ lower rectangles

$\cup(\sigma, f)$ upper rectangles

Example 4

Let $f(x) = 5 + 2x$, $f: [1, 4] \rightarrow IR$ Find $L(\sigma, f)$ and $\cup(\sigma, f)$ using 3 partitions.

Solution

We divide the interval [1, 4] into 3 subintervals.

$$h = \frac{b-a}{n} = \frac{4-1}{3} = 1 \qquad \Rightarrow \sigma = (1,2,3,4)$$

The intervals are [1, 2], [2, 3], [3, 4], $f(x) = 5 + 2x \Rightarrow f'(x) = 2 > 0$

∴ there is no critical point and function is increasing in its domain. We can find the value of $L(\sigma, f)$, $\cup(\sigma, f)$ using the following table where m_i is the smallest values, M_i is the greatest .

Subintervals [a, b]	length of intervals h	m_i	M_i	$h_i m_i$	$h_i M_i$
[1, 2]	1	$m_1 = 5 + 2 = 7$	$M_1 = 5 + 4 = 9$	7	9
[2, 3]	1	$m_2 = 5 + 4 = 9$	$M_2 = 5 + 6 = 11$	9	11
[3, 4]	1	$m_3 = 5 + 6 = 11$	$M_3 = 5 + 8 = 13$	11	13
				$\sum h_i m_i = 27$	$\sum h_i M_i = 33$

$\therefore \sum h_i m_i = L(\sigma, f) = 27, \ \sum h_i M_i = \cup(\sigma, f) = 33$

Example 5

If $f(x) = 3x - x^2$ $f: [0, 4] \rightarrow IR$, find each of $L(\sigma, f)$ and $\cup(\sigma, f)$ by using 4 equal partitions.

Solution

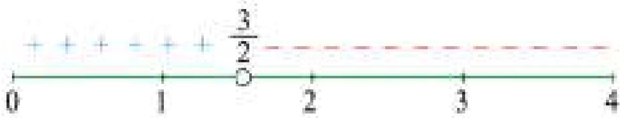
$$h = \frac{b-a}{n} = \frac{4-0}{4} = 1$$

$\Rightarrow \sigma = (0, 1, 2, 3, 4)$

[0, 1], [1, 2], [2, 3], [3, 4]

$f(x) = 3x - x^2 \Rightarrow f'(x) = 3 - 2x$

$f'(x) = 0 \Rightarrow x = \frac{3}{2} \in [1, 2]$, so critical point is on interval [1, 2]



sing chart of $f'(x)$

Subintervals [a, b]	length of intervals h	m_i	M_i	$h_i m_i$	$h_i M_i$
[0, 1]	1	0	2	0	2
[1, 2]	1	2	$\frac{9}{4}$	2	$\frac{9}{4}$
[2, 3]	1	0	2	0	2
[3, 4]	1	-4	0	-4	0
				$\sum h_i m_i = -2$	$\sum h_i M_i = 6\frac{1}{4}$

$\therefore \sum h_i m_i = L(\sigma, f) = -2, \ \sum h_i M_i = \cup(\sigma, f) = 6\frac{1}{4}$

note that : $L(\sigma, f) \le \cup(\sigma, f)$

Exercises

Find $U(\sigma,f)$, $L(\sigma,f)$ for the following

1) $f:[-2,1]\to R$, $f(x)=3-x$

 a) $\sigma=(-2,0,1)$

 b) The interval [-2, 1] is divided into three regular sub-intervals

2) if $f:[0,4]\to R$, $f(x)=4x-x^2$ if $\sigma=(0,1,2,3,4)$

3) $f:[1,4]\to R$, $f(x)=3x^2+2x$

 a) $\sigma=(1,2,4)$

 b) Use three regular partitions.

4 - 3 Definition of Integration

Theorem 4-3-1

If $f:[a,b]\to IR$ is continuous on [a, b] and $L(\sigma,f)\le U(\sigma,f)$ then there is a certain "k" where

$L(\sigma,f)\le k\le U(\sigma,f)$ for each partition σ to the interval [a , b]

We call the number k the definite integral for the function f on [a, b] and show by

$$\int_a^b f(x)\,dx$$

and read as integral of f from a to b and a, b are called limits of integral.

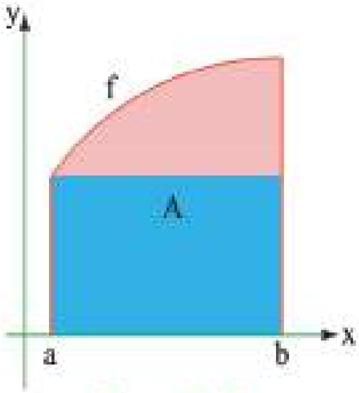
Notes

1) If f is continuous on [a, b] then $L(\sigma,f)\le \int_a^b f(x)\,dx\le U(\sigma,f)$ so, the approximate value of the integral is

$$\int_a^b f(x)\,dx \approx \frac{L(\sigma,f)+U(\sigma,f)}{2}$$

2) If $\forall x \in [a, b], f(x) \geq 0$ then $\int_a^b f(x)\,dx$ gives the area of the region A. under the curve f whic nonnegative number.

Here dx means (indicates) that the values of a, b are the values of the variable x.



(Figure 4-19)

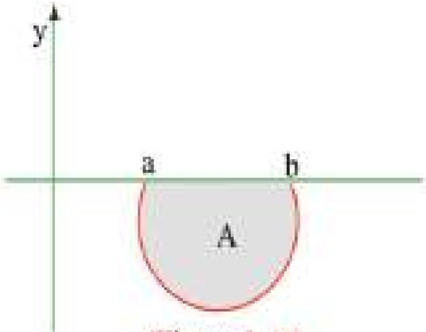
3) If $f(x) \leq 0, \forall x \in [a, b]$ then

$$\int_a^b f(x)\,dx \leq 0$$

And this doesn't mean the area,but the area of the region

A shown in Figure 4-20 is

$$-\int_a^b f(x)\,dx = \left| \int_a^b f(x)\,dx \right|$$



(Figure 4-20)

4) The value of $\int_a^b f(x)dx$ depends on the interval [a, b] and the function f(x).

Example 1

Let $f(x) = x^2$, $f: [1, 3] \to IR$ then find the approximate vaule of $\int_1^3 x^2 dx$ if the interval [1, 3] is divided into two sub intervals.

Solution

$f(x) = x^2$, f is continuous on [1, 3] since it is polynomial function.

$\therefore f'(x) = 2x, f'(x) = 0 \Rightarrow 0 = 2x \Rightarrow x = 0$ is critical number but $0 \notin [1, 3]$

$$h = \frac{b-a}{n} = \frac{3-1}{2} = 1$$

[a, b] (subintervals)	b - a = (h)	$h_i m_i$	$h_i M_i$
[1, 2]	1	1	4
[2, 3]	1	4	9

$\therefore$ The local maximum and local minimum of each intervals will be on the end points of each interval [1 , 2] , [2 , 3]

$L(\sigma, f) = 1\,(1) + 1\,(4) = 1 + 4 = 5$

$U(\sigma, f) = 1(4) + 1(9) = 4 + 9 = 13$

$$\therefore \int_1^3 x^2 dx = \frac{5+13}{2} = 9 \quad \text{approximatelly}$$

Example 2

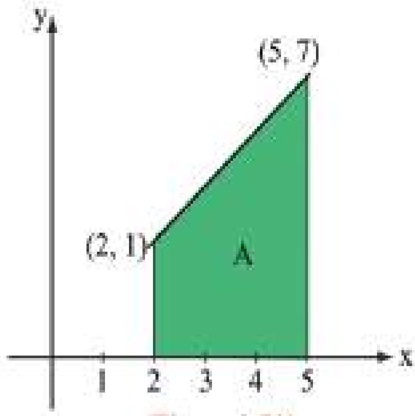
If f: [2, 5] → IR, f(x) = 2x – 3 then find $\int_2^5 f(x).dx$

Solution

Notice that f(x) > 0, ∀x∈[2, 5]

∴ the integral $\int_2^5 f(x).dx$ will be area of A. and it is trapezoid region.

∴ the area of region A is



(Figure 4-21)

$$A = \frac{1}{2} \text{ (sum of two parallel bases)} \times \text{ height}$$

$$\therefore \quad A = \frac{1}{2} \, . \, [1 + 7] \, (3) = \frac{1}{2} \, (8) \, (3) = 12 \text{ unit}^2$$

$$\therefore \quad \int_2^5 f(x)dx = 12$$

It can be also found by the previous way as follow.

Subintervals [a, b]	length of intervals $h_i = b - a$	M_i	m_i	$h_i M_i$	$h_i m_i$
[2, 3]	1	3	1	3	1
[3, 5]	2	7	3	14	6
				$\sum h_i M_i = 17$	$\sum h_i m_i = 7$

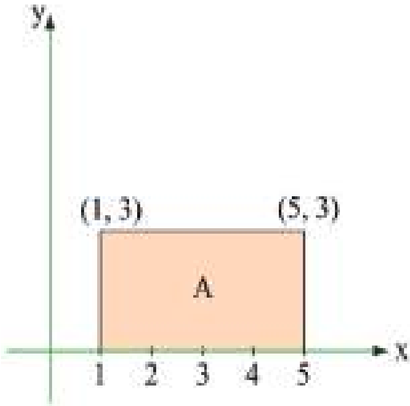
$$\int_2^5 (2x-3)dx = \frac{17+7}{2} = \frac{24}{2} = 12 \quad \text{unit}^2$$

Example 3

If $f(x) = 3$, $f: [1, 5] \rightarrow IR$ then find $\int_1^5 f(x)dx$

Solution

In the figure 4-22 notice that the region of A is rectangular region whose base is 5-1 = 4 units and width is 3 units



(Figure 4-22)

$\therefore \quad A = 4(3) = 12 \text{ unit}^2$

$$\therefore \int_1^5 f(x)dx = 12 \text{ unit}^2$$

the second way

Subintervals [a, b]	length of intervals $h_i = b - a$	M_i	m_i	$h_i m_i$	$h_i M_i$
[1, 3]	2	3	3	6	6
[3, 5]	2	3	3	6	6
				$\sum h_i m_i = 12$	$\sum h_i M_i = 12$

$L(\sigma, f) = \sum h_i m_i = 12, \ U(\sigma, f) = \sum h_i M_i = 12$

$$\therefore \int_1^5 3dx = \frac{12 + 12}{2} = \frac{24}{2} = 12 \quad \text{unit}^2$$

Exercises

1) Find approximate value for integration $\int_1^3 \frac{3}{x}dx$ using partition $\sigma = (1,2,3)$.

2) Let $f(x) = 3x - 3$, $f:[1,4] \to R$, Find value of integration $\int_1^4 f(x)dx$ using the partition, $\sigma = (1,2,3,4)$ and then satisfy geometrically by calculating area region under the curve f.

3) Find approximate value of integration $\int_2^4 (3x^2 - 3)dx$ using the partition $\sigma = (2,3,4)$

4) Find value of integration $\int_{-3}^2 f(x)dx$ such that f(x)= -4

5) Find approximate value of integration $\int_1^5 x^3 dx$ using four regular partitions.

4 - 4 The Fundomental Theorem of Integration

We learned in the previous section finding the value of integral $\int_a^b f(x)dx$ such that f is continuous on [a, b] by using the area.

The following theorem helps us to finding the definite integral.

Theorem 4-4-1

If f is continuous on [a, b] then there exist a continuous function F on [a, b] which is called "Anti-derivative" of the function f such that

$$\int_a^b f(x)dx = F(b) - F(a) \quad \text{where } F'(x) = f(x),\ \forall x \in (a, b)$$

For examle,

If $f(x) = 2x$, $f : [1, 2] \to IR$ then $F(x) = x^2$, $F:[1, 2] \to IR$

$\acute{F}(x) = 2x = f(x), \forall x \in (1, 2)$, **and upon it**

$$\int_1^2 f(x)dx = F(2) - F(1)$$
$$= 4 - 1 = 3$$

Note

$F(2) - F(1)$ is written in the form $\left[F(x)\right]_1^2$

Example 1

If $f(x)$ is a continuous function on $[1, 5]$ such that $F(x) = 3x^2$ is antiderivative of f then find

$F[1, 5] \to IR$ $\int_1^5 f(x)dx$

Solution

$$\int_1^5 f(x)dx = F(5) - F(1) = 3\ (5)^2 - 3\ \ (1)^2 = 75 - 3 = 72$$

and we can write that as following

$$\int_1^5 f(x)dx = \left[F(x)\right]_1^5 = \left[3x^2\right]_1^5 = 75 - 3 = 72$$

Example 2

If f is a continuous function on $\left[0, \frac{\pi}{2}\right]$ and the antiderivative of f is $F(x) = \sin x$,

$F:\left[0, \frac{\pi}{2}\right] \to IR$ then find $\int_0^{\pi/2} f(x)dx$.

Solution

$$\int_0^{\pi/2} f(x)\,dx = \left[F(x)\right]_0^{\pi/2} = F\left(\frac{\pi}{2}\right) - F(0) = \sin\frac{\pi}{2} - \sin 0 = 1 - 0 = 1$$

Example 3

Prove that $F : [1, 3] \rightarrow IR$, $F(x) = x^3 + 2$ is an antiderivative of the function $f(x) = 3x^2$

Solution

$F(x) = x^3 + 2$ is a continuous and differentiable on IR since it is polynamial function.

$\therefore$ F is continuous on $[1, 3]$ and differentiable on $(1, 3)$

$\therefore$ $F'(x) = 3x^2 = f(x), \forall x \in (1, 3)$

$\therefore$ F is antiderivative of f on $[1, 3]$

Example 4

Prove that F : $F(x) = \frac{1}{2}\sin 2x$ is an antiderivative of the function $f : R \rightarrow R$, $f(x) = \cos 2x$

then find $\int_0^{\pi/4} \cos 2x.dx$.

Solution

$f(x) = \cos 2x$, $f : R \rightarrow [-1,1]$ is a continuous and differentiable on IR since it is trigonometric function defined on it's domain .

$F(x) = \frac{1}{2}\sin 2x$ is a continuous and differentiable on IR

$F'(x) = \frac{1}{2}\cos 2x.(2) = \cos 2x = f(x) \qquad \forall x \in IR$

F is antiderivative of f function

$$\int_a^b f(x)dx = F(b) - F(a) \qquad \text{(Theorem 4 - 2)}$$

$$\int_0^{\pi/4} \cos 2x dx = \left[\frac{1}{2}\sin 2x\right]_{x=0}^{x=\frac{\pi}{4}} = \frac{1}{2}\sin\left(2.\frac{\pi}{4}\right) - \frac{1}{2}\sin(2)(0)$$

$$= \frac{1}{2}\sin\frac{\pi}{2} - \frac{1}{2}\sin 0 = \frac{1}{2}(1) - 0 = \frac{1}{2}$$

The following table shows the function f and its antiderivative F.

f(x)	F(x)
a	ax
x^n, $n \neq -1$	$\frac{x^{n+1}}{n+1}$
ax^n, $n \neq -1$	$\frac{ax^{n+1}}{n+1}$
$[f(x)]^n . f'(x)$, $n \neq -1$	$\frac{[f(x)]^{n+1}}{n+1}$
sin(ax+b)	$-\frac{1}{a}\cos(ax+b)$
cos(ax+b)	$\frac{1}{a}\sin(ax+b)$
$\sec^2(ax+b)$	$\frac{1}{a}\tan(ax+b)$
$\csc^2(ax+b)$	$-\frac{1}{a}\cot(ax+b)$
sec ax tan ax	$\frac{1}{a}\sec ax$
csc ax cot ax	$-\frac{1}{a}\csc ax$

In the table antiderivative of f is F+c where c is constant real number.

Example 5

Find $\int_{0}^{\frac{\pi}{4}} \sec^2 x dx$

Solution

$$\int_{0}^{\frac{\pi}{4}} \sec^2 x dx = \left[\tan x\right]_{0}^{\frac{\pi}{4}} = \tan\frac{\pi}{4} - \tan 0 = 1 - 0 = 1$$

Example 6

Find $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \csc^2 x \, dx$

Solution

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \csc^2 x \, dx = \left[-\cot x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = -\cot\frac{\pi}{2} + \cot\frac{\pi}{4} = 0 + 1 = 1$$

Example 7

Find $\int_{0}^{\frac{\pi}{3}} \sec x \tan x \, dx$

Solution

$$\int_0^{\frac{\pi}{3}} \sec x \ \tan x \ dx = \left[\sec x\right]_0^{\frac{\pi}{3}} = \sec\frac{\pi}{3} - \sec 0 = 2 - 1 = 1$$

Example 8

Find $\int_1^3 x^3 dx$

Solution

$$\int_1^3 x^3 dx = \left[\frac{x^4}{4}\right]_1^3 = \frac{3^4}{4} - \frac{1}{4} = \frac{81}{4} - \frac{1}{4} = \frac{80}{4} = 20$$

4 - 5 Properties of Definite Integral

Firstly,

1. f is continuous on [a, b] and if

$$f(x) \geq 0, \ \forall x \in [a, b] \ \Rightarrow \int_a^b f(x)dx \geq 0$$

For example,

a) $\int_{-1}^{2} x^2 dx \geq 0$ since $f(x) = x^2 \geq 0, \ \forall x \in [-1, 2]$

b) $\int_{-2}^{3} 3dx > 0$ since $f(x) = 3 > 0, \forall x \in [-2, 3]$

c) $\int_{2}^{3} (x+1)dx > 0$ since $f(x) = (x+1) > 0, \forall x \in [2, 3]$

2. f is continuous on [a, b] and if $\forall x \in [a, b], f(x) \leq 0$ then $\int_{a}^{b} f(x)dx \leq 0$

For example

a) $\int_{1}^{2} (-2)dx < 0$ since $f(x) < 0, \forall x \in [1, 2]$

b) $\int_{-2}^{-1} x\,dx < 0$ since $f(x) \leq 0, \forall x \in [-2, -1]$

Secondly,

f is continuous on [a, b],c is a constant real number then

$$\int_{a}^{b} c\,f(x)dx = c\int_{a}^{b} f(x)dx$$

Example 9

If $\int_{2}^{5} f(x)dx = 8$ then find $\int_{2}^{5} 5f(x)dx$

Solution

$$\int_{2}^{5} 5f(x)dx = 5\int_{2}^{5} f(x)dx = 5\,(8) = 40$$

Thirdly,

If $f_1(x)$, $f_2(x)$ are two continuous functions on [a, b] then

$$\int_a^b \left(f_1(x) \mp f_2(x)\right)dx = \int_a^b f_1(x)\,dx \mp \int_a^b f_2(x)\,dx$$

We can apply this property for sum of many functions which are continuous on [a, b].

Example 10

If $\int_1^3 f_1(x)dx = 15$, $\int_1^3 f_2(x)dx = 17$ then find each of the following :

$$\int_1^3 \left(f_1(x) + f_2(x)\right)dx, \quad \int_1^3 \left(f_1(x) - f_2(x)\right)dx$$

Solution

$$\int_1^3 \left(f_1(x) + f_2(x)\right)dx = \int_1^3 f_1(x)\,dx + \int_1^3 f_2(x)\,dx = 15 + 17 = 32$$

$$\int_1^3 \left(f_1(x) - f_2(x)\right)dx = \int_1^3 f_1(x)\,dx - \int_1^3 f_2(x)\,dx = 15 - 17 = -2$$

Example 11

If $f(x) = 3x^2 + 2x$ then find $\int_1^2 f(x)\,dx$

Solution

$$\int_1^2 f(x)\,dx = \int_1^2 (3x^2+2x)\,dx = \int_1^2 3x^2 dx + \int_1^2 2x\,dx$$

$$= \left[x^3\right]_1^2 + \left[x^2\right]_1^2 = (8-1)+(4-1) = 7+3 = 10$$

Fourthly,

If f(x) is continuous on [a, b] and $c \in (a, b)$ then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Example 12

If $\int_1^3 f(x)\,dx = 5$, $\int_3^7 f(x)\,dx = 8$ then find $\int_1^7 f(x)\,dx$

Solution

$$\int_1^7 f(x)\,dx = \int_1^3 f(x)\,dx + \int_3^7 f(x)\,dx = 5+8 = 13$$

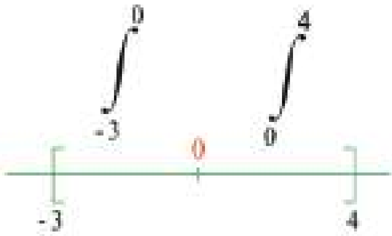
Example 13

If $f(x) = |x|$ then find $\int_{-3}^4 f(x)\,dx$

Solution

If $f(x)$ is continious on [–3, 4] and the absolute vaule function as the rule,

$$f(x)=\begin{cases} x, & \forall x \ge 0 \\ -x, & \forall x < 0 \end{cases}$$



$$\int_{-3}^{4} f(x)\,dx = \int_{-3}^{0} (-x)\,dx + \int_{0}^{4} x\,dx = \left[\frac{-x^2}{2}\right]_{-3}^{0} + \left[\frac{x^2}{2}\right]_{0}^{4}$$

$$= \left[0+\frac{9}{2}\right] + \left[\frac{16}{2}-0\right] = \frac{9}{2}+\frac{16}{2} = \frac{25}{2}$$

Example 14

If $f(x)=\begin{cases} 2x+1, & \forall x \ge 1 \\ 3, & \forall x < 1 \end{cases}$ then find $\int_{0}^{5} f(x)\,dx$

Solution

f is continious on [0, 5]

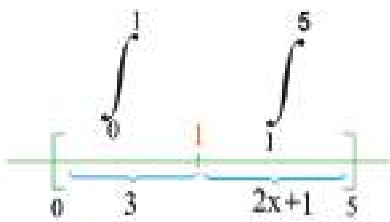
(i) $f(1) = 2(1) + 1 = 3$

(ii) $\lim_{x\to 1} f(x) = \begin{cases} \lim_{x\to 1}(2x+1) = 3 = L_1 \\ \lim_{x\to 1} 3 = 3 = L_2 \end{cases}$ $\quad L_1 = L_2$

$$\lim_{x\to 1} f(x) = 3 \Rightarrow \lim_{x\to 1} f(x) = f(1)$$

And also f is continuous on $\{x : x > 1\}$, $\{x : x < 1\}$ so it is continuous on [0, 5]

$$\int_{0}^{5} f(x)\,dx = \int_{0}^{1} f(x)\,dx + \int_{1}^{5} f(x)\,dx$$



$$= \int_{0}^{1} 3\,dx + \int_{1}^{5} (2x+1)\,dx = \left[3x\right]_{0}^{1} + \left[x^2+x\right]_{1}^{5}$$

$$= \left[3-0\right] + \left[25+5\right] - \left[2\right] = 3+28 = 31$$

Fifthly,

a) $\int_a^a f(x)dx = 0$ b) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

For example,

a) $\int_3^3 xdx \Rightarrow \left[\frac{x^2}{2}\right]_3^3 = \frac{9}{2} - \frac{9}{2} = 0$

using the rule.

$\int_3^3 xdx = 0$

b) $\int_3^2 3x^2dx = -\int_2^3 3x^2dx$

$= -\left[x^3\right]_2^3$

$= -\left[27\right] + \left[8\right] = -19$

Exercises

1) Calculate the following integrations

a) $\int_{-2}^{2} (3x-2)\,dx$

b) $\int_1^2 \left(x^{-2} + 2x + 1\right)dx$

c) $\int_1^3 \left(x^4 + 4x\right)dx$

d) $\int_0^2 |x-1|dx$

e) $\int_{-\frac{\pi}{2}}^{0} (x + \cos x)\,dx$

f) $\int_3^2 \frac{x^3-1}{x-1}\,dx$

g) $\int_1^3 \frac{2x^3 - 4x^2 + 5}{x^2}dx$

2) Prove that F(x)=sinx+x is anti-derivative of function f(x) whereby $F:\left[0,\frac{\pi}{6}\right]\to R$

$f(x)=1+\cos x$ Whereby $f:\left[0,\frac{\pi}{6}\right]\to R$ then compute $\int_0^{\frac{\pi}{6}} f(x)dx$

3) Find the following integrations

a) $\int_1^4 (x-2)(x+1)^2 dx$

b) $\int_{-1}^1 |x+1| dx$

c) $\int_2^3 \frac{x^4-1}{x-1} dx$

d) $\int_0^1 \sqrt{x}\left(\sqrt{x}+2\right)^2 dx$

4) If $f(x)=\begin{cases}2x, & \forall x\geq 3\\ 6, & \forall x<3\end{cases}$ then find $\int_1^4 f(x)dx$

5) If $f(x)=\begin{cases}3x^2, & \forall x\geq 0\\ 2x, & \forall x<0\end{cases}$ then find $\int_{-1}^3 f(x)dx$

4 - 6 Indefinite Integral

1. $F_1 : [1, 3] \to R, F_1(x) = x^2 + 1$
2. $F_2 : [1, 3] \to R, F_2(x) = x^2 + \frac{1}{2}$
3. $F_3 : [1, 3] \to R, F_3(x) = x^2 - \sqrt{2}$
4. $F_4 : [1, 3] \to R, F_4(x) = x^2 - 5$

Note

Note That F_1, F_2, F_3, F_4 stisfy:

i) Continuous on $[1,3]$.

ii) Differentiable on $(1, 3)$

iii) $\forall x \in (1, 3), F_1' = F_2' = F_3' = F_4' = 2x$ and so

$$F_1(x) - F_2(x) = (x^2 + 1) - (x^2 + \frac{1}{2}) = \frac{1}{2}$$

$$F_1(x) - F_4(x) = (x^2 + 1) - (x^2 - 5) = 6$$

So : In general $\int f(x).dx = F(x) + C$

where c is arbitrary real constant

Example 1

Find $\int f(x).dx$ if you know that

a) $f(x) = 3x^2 + 2x + 1$

b) $f(x) = \cos x + x^{-2}$

c) $f(x) = x + \sec x \tan x$

d) $f(x) = \sin (2x + 4)$

Solution

a) $$\int (3x^2 + 2x + 1)dx = \frac{3x^3}{3} + \frac{2x^2}{2} + x + c = x^3 + x^2 + x + c$$

b) $$\int (\cos x + x^{-2})dx = \sin x + \frac{x^{-1}}{-1} + c = \sin x - \frac{1}{x} + c$$

c) $$\int (x+\sec x.\tan x)dx = \frac{x^2}{2} + \sec x + c$$

d) $$\int \sin(2x+4)dx = \frac{-1}{2}\cos(2x+4) + c$$

Example 2

Find each of the following integrals.

a) $$\int (x^2+3)^2 (2x)\, dx$$

c) $$\int \sin^4 x \cos x dx$$

b) $$\int (3x^2+8x+5)^6 .(3x+4)dx$$

d) $$\int \tan^6 x \sec^2 x dx$$

Solution

a) $$\int (x^2+3)^2 (2x)\, dx$$

We assume that $f(x) = x^2 + 3$ then $f'(x) = 2x$

$$\int (x^2+3)^2(2x)dx = \int \left[f(x)\right]^2 f'(x)\, dx = \frac{1}{3}\left[f(x)\right]^3 + c = \frac{1}{3}(x^2+3)^3 + c$$

b) $$\int (3x^2+8x+5)^6 .(3x+4)dx$$

We assume that

$f(x) = 3x^2 + 8x + 5$ then $f'(x) = 6x + 8$

$$\int (3x^2+8x+5)^6 .(3x+4)dx = \frac{1}{2}\int (3x^2+8x+5)^6 .(6x+8)dx$$

$$= \frac{1}{2}\int \left[f(x)\right]^6 f'(x)\, dx = \frac{1}{2}\cdot\frac{\left[f(x)\right]^7}{7} + c = \frac{1}{14}(3x^2+8x+5)^7 + c$$

c) $\int \sin^4 x \cos x dx$

We assume that

$f(x) = \sin x \Rightarrow f'(x) = \cos x$

$$\int \sin^4 x \cos x dx = \int \left[f(x)\right]^4 . f'(x) dx = \frac{\left[f(x)\right]^5}{5} + c = \frac{1}{5}\sin^5 x + c$$

d) $\int \tan^6 x \sec^2 x dx$

We assume that

$f(x) = \tan x \Rightarrow f'(x) = \sec^2 x$

$$\therefore \int \tan^6 x \sec^2 x dx = \int \left[f(x)\right]^6 . f'(x) dx = \frac{\left[f(x)\right]^7}{7} + c = \frac{1}{7}\tan^7 x + c$$

4-6-1 Integral of Square of the Trigonometric Functions

Here we can also remaind these to stadents $\sin^2\theta + \cos^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta$

1. $\int \sec^2\theta d\theta = \tan\theta + c$
2. $\int \csc^2\theta d\theta = -\cot\theta + c$
3. $\int \tan^2\theta \, d\theta = \int (\sec^2\theta - 1) d\theta = \int \sec^2\theta d\theta - \int d\theta = \tan\theta - \theta + c$
4. $\int \cot^2\theta \, d\theta = \int (\csc^2\theta - 1) d\theta = -\cot\theta - \theta + c$
5. $\int \sin^2\theta \, d\theta = \int \frac{1 - \cos 2\theta}{2} d\theta = \frac{1}{2}\int d\theta - \frac{1}{4}\int \cos 2\theta (2) d\theta$

   $= \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + c$
6. $\int \cos^2\theta \, d\theta = \int \frac{1 + \cos 2\theta}{2} d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + c$

Examples

Find each of the following integrales

1. $\int 9\sin 3x dx = 3\int 3\sin 3x dx = -3\cos 3x + c$

2. $\int x^2 \sin x^3 dx = \frac{1}{3}\int \sin x^3 \ (3x^2)\ dx = -\frac{1}{3}\cos x^3 + c$

3. $\int \sqrt{1-\sin 2x} dx = \int \sqrt{\sin^2 x - 2\sin x\cos x + \cos^2 x} dx = \sqrt{(\sin x - \cos x)^2} dx$

$$= \pm\int (\sin x - \cos x) dx = \pm(\cos x + \sin x) + c$$

4. $\int \sin^4 x\ dx = \int \frac{(1-\cos 2x)^2}{4} dx = \frac{1}{4}\int dx - \frac{1}{4}\int 2\cos 2x dx + \frac{1}{4}\int \cos^2 2x dx$

$$= \frac{1}{4}\int dx - \frac{1}{4}\int 2\cos 2x dx + \frac{1}{8}\int dx + \frac{1}{32}\int 4\cos 4x dx$$

$$= \frac{1}{4}x - \frac{1}{4}\sin 2x + \frac{1}{8}x + \frac{1}{32}\sin 4x + c = \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c$$

5. $\int (\sin x - \cos x)^7 (\cos x + \sin x) dx = \frac{(\sin x - \cos x)^8}{8} + c$

6. $\int \frac{1+\tan^2 x}{\tan^3 x} dx = \int \tan^{-3} x \sec^2 x dx = \frac{\tan^{-2} x}{-2} + c = \frac{-1}{2\tan^2 x} + c$

7. $\int \cos^3 x dx = \int \cos x(1-\sin^2 x) dx = \int \cos x dx - \int \sin^2 x \cos x dx = \sin x - \frac{\sin^3 x}{3} + c$

8. $\int \frac{\tan x}{\cos^2 x} dx = \int \tan x \sec^2 x dx = \frac{\tan^2 x}{2} + c$

9. $\int \sin 6x \cos^2 3x dx = \int (2 \sin 3x \cos 3x) \cos^2 3x dx = 2 \int \cos^3 3x \sin 3x dx$

$= (\frac{-2}{3}) \times \frac{\cos^4 3x}{4} + c = -\frac{1}{6} \cos^4 3x + c$

10. $\int \frac{\cos 4x}{\cos 2x - \sin 2x} dx = \int \frac{\cos^2 2x - \sin^2 2x}{\cos 2x - \sin 2x} dx = \int \frac{(\cos 2x - \sin 2x)(\cos 2x + \sin 2x)}{\cos 2x - \sin 2x} dx$

$= \int (\cos 2x + \sin 2) dx = \frac{1}{2} \sin 2x - \frac{1}{2} \cos 2x + c$

11. $\int \sin^2 3x dx = \frac{1}{2} x - \frac{1}{12} \sin 6x + c$

12. $\int \cot^2 5x dx = -\frac{1}{5} \cot 5x - x + c$

13. $\int \tan^2 7x dx = \frac{1}{7} \tan 7x - x + c$

Exercises

Calculate the following integrations.

1. $\int \frac{(2x^2-3)^2-9}{x^2} dx$

2. $\int \frac{(3-\sqrt{5x})^7}{\sqrt{7x}} dx$

3. $\int \frac{\cos^3 x}{1-\sin x} dx$

4. $\int \csc^2 x \cos x \, dx$

5. $\int \frac{x}{(3x^2+5)^4} dx$

6. $\int \sqrt[3]{x^2+10x+25} \, dx$

7. $\int \sin^3 x\,dx$

8. $\int \frac{\cos\sqrt{1-x}}{\sqrt{1-x}}dx$

9. $\int (3x^2+1)^2 dx$

10. $\int \frac{\sqrt{\sqrt{x}-x}}{\sqrt[4]{x^3}}\,dx$

11. $\int (1+\cos 3x)^2 dx$

12. $\int \sec^2 4x\,dx$

13. $\int \csc^2 2x\,dx$

14. $\int \tan^2 8x\,dx$

15. $\int \frac{\sqrt{\cot 2x}}{1-\cos^2 2x}dx$

16. $\int \cos^2 2x\,dx$

17. $\int \sin^2 8x\,dx$

18. $\int \cos^4 3x\,dx$

4 - 7 The Natural Logarithm

Definition 4-7-1

$$\ln x = \int_1^x \frac{1}{t}dt, \quad \forall x > 0$$

i) if $x = 1$

$$\ln 1 = \int_1^1 \frac{1}{t}dt = 0 \qquad \left(\int_a^a f(x).dx = 0\right)$$

ii) if $0 < x < 1$

$$\ln x = \int_1^x \frac{1}{t}dt = -\int_x^1 \frac{1}{t}dt$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

By using chain Rule :

$$\frac{d}{dx}(\ln u) = \frac{d(\ln u)}{du}.\frac{du}{dx}$$

$$\therefore \frac{d}{dx}(\ln u) = \frac{1}{u}.\frac{du}{dx} \Rightarrow d(\ln u) = \frac{1}{u}du.$$

04 INTEGRATION

Example 1

If $y = \ln(3x^2 + 4)$ then find $\frac{dy}{dx}$.

Solution

$$\frac{dy}{dx} = \frac{1}{3x^2+4} \cdot \frac{d(3x^2+4)}{dx}$$

$$= \frac{6x}{3x^2+4}$$

$$d(\ln u) = \frac{1}{u}du \quad \Rightarrow \quad \int \frac{du}{u} = \ln u + c \qquad \text{Where} \qquad u > 0 \text{ ,}$$

Example 2

Find $\int \frac{\cos\theta d\theta}{1+\sin\theta}$

Solution

Assume that $u = 1 + \sin\theta$

$$\frac{du}{d\theta} = \cos\theta \Rightarrow du = \cos\theta \, d\theta$$

$$\therefore \quad \int \frac{\cos\theta d\theta}{1+\sin\theta} = \int \frac{du}{u} = \ln|u| + c$$

$$= \ln|1 + \sin\theta| + c$$

4-7-2 The Natural Logaritmic Function

Let $y = \ln x$ $\quad \{(x, y) : y = \ln x, x > 0\}$

$x = \ln^{-1}(y), \quad y > 0, \quad x \in IR$

$x = e^{y}$

Domain of $\ln^{-1}(y)$ is the range of $\ln(x)$

Conclusion

Exponantial function e^{x} (e base) is the inverse of natural logaritmic function ,it derives all of its properties from this fact

Theorem 4-7-3

$$\frac{d}{dx}(e)^{x} = e^{x}$$

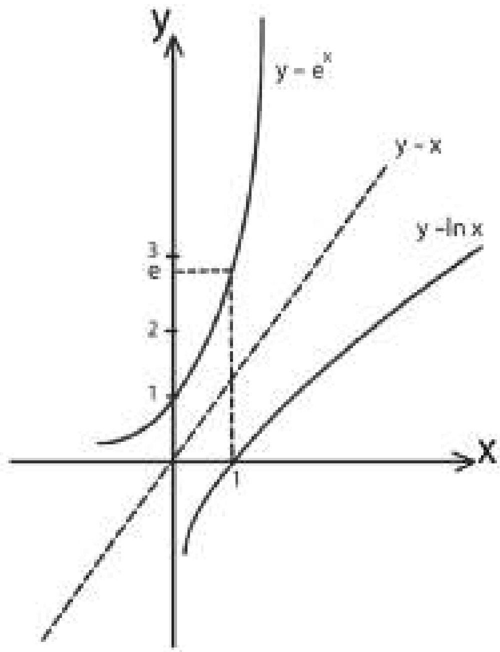
Proof : $y = e^{x}$

$x = \ln y \Rightarrow$

$1 = \frac{1}{y} \cdot \frac{dy}{dx} \Rightarrow$

$\frac{dy}{dx} = y = e^{x}$

$$\frac{d}{dx}\left(e^{u}\right) = e^{u} \cdot \frac{du}{dx}$$



Example 3

If $y = e^{\tan x}$ then find $\frac{dy}{dx}$.

Solution

$$\frac{d(e^{\tan x})}{dx} = e^{\tan x}.\frac{d(\tan x)}{dx} \Rightarrow \frac{dy}{dx} = e^{\tan x}.\sec^2 x$$

Note

$$d(e^u) = e^u \frac{du}{dx} \quad \Rightarrow$$

$$\int e^u du = e^u + c$$

Example 4

Find $\int x.e^{x^2} dx$

Solution

$x^2 = u \Rightarrow 2xdx = du$

$$\therefore \int e^{x^2} xdx = \frac{1}{2}\int e^u \, du = \frac{1}{2} e^u + c$$

$$\frac{1}{2} e^{x^2} + c$$

Definition 4-7-4

If a is a positive number then

$a^u = e^{u.\ln a}$

Theorem 4-3

$$\frac{da^u}{dx} = a^u.\frac{du}{dx}.\ln a$$

Proof:

$$\frac{da^u}{dx} = \frac{d(e^{u\ln a})}{dx}$$

$$= e^{u\ln a}.\frac{d}{dx}(u\ln a)$$

$$\frac{da^u}{dx} = a^u.\frac{du}{dx}.\ln a \qquad \text{(Q. E. D)}$$

Example 5

Find $\frac{dy}{dx}$ for each of the followings.

a) $y = 3^{2x-5}$ b) $y = 2^{-x^2}$ c) $y = 5^{\sin x}$

Solution

a) $y = 3^{2x-5} \Rightarrow \frac{dy}{dx} = 3^{2x-5}\ (2).\ln 3$

$= (2\ln 3)3^{2x-5}$

b) $y = 2^{-x^2} \Rightarrow \frac{dy}{dx} = 2^{-x^2}(-2x)\ln 2$

$= (-2x\ln 2)\ (2^{-x^2})$

c) $y = 5^{\sin x} \Rightarrow \frac{dy}{dx} = 5^{\sin x}.\cos x(\ln 5)$

$= (\ln 5).5^{\sin x}.\cos x$

Exercises

1) Find $\frac{dy}{dx}$ for the following.

a) $y = \ln 3x$

b) $y = \ln\left(\frac{x}{2}\right)$

c) $y = \ln(x^2)$

d) $y = (\ln x)^2$

e) $y = \ln\left(\frac{1}{x}\right)^3$

f) $y = \ln(2 - \cos x)$

g) $y = e^{-5x^2+3x+5}$

h) $y = 9^{\sqrt{x}}$

i) $y = 7^{\frac{-x}{4}}$

j) $y = x^2 e^x$

2) Find the following integrations.

a) $\int_0^3 \frac{1}{x+1} dx$

b) $\int_0^4 \frac{2x}{x^2+9} dx$

c) $\int_{\ln 3}^{\ln 5} e^{2x} dx$

d) $\int_0^{\ln 2} e^{-x} dx$

e) $\int_0^1 (1+e^x)^2 e^x dx$

f) $\int_0^1 \frac{3x^2+4}{x^3+4x+1} dx$

g) $\int_1^4 \frac{e^{\sqrt{x}} dx}{2\sqrt{x}}$

h) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2 x}{2+\tan x} dx$

i) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}} dx$

j) $\int \cot^3 5x \, dx$

k) $\int_{0}^{2} e^{\cos x} \sin x \, dx$

l) $\int_{1}^{2} x e^{-\ln x} \, dx$

3) Prove that:

a) $\int_{1}^{8} \frac{\sqrt{\sqrt[3]{x}-1}}{\sqrt[3]{x^2}} dx = 2$

b) $\int_{-2}^{4} |3x-6| \, dx = 30$

4) $f(x)$ is a continuous function on interval $[-2,6]$, if $\int_{1}^{6} f(x) \, dx = 6$ and $\int_{-2}^{6} [f(x)+3] \, dx = 32$

then find $\int_{-2}^{1} f(x) \, dx$

5) Find value of $a \in \mathbb{R}$ if $\int_{1}^{a} \left(x + \frac{1}{2} \right) dx = 2 \int_{0}^{\frac{\pi}{4}} \sec^2 x \, dx$

6) Let $f(x) = x^2 + 2x + k$, where by $k \in \mathbb{R}$, a function its min. is (-5), then find $\int_{1}^{3} f(x) \, dx$

7) If the curve $f(x) = (x-3)^3 + 1$ has inflection point (a, b), then find the numerical value of :

$\int_{0}^{b} f'(x) \, dx - \int_{0}^{a} f''(x) \, dx$

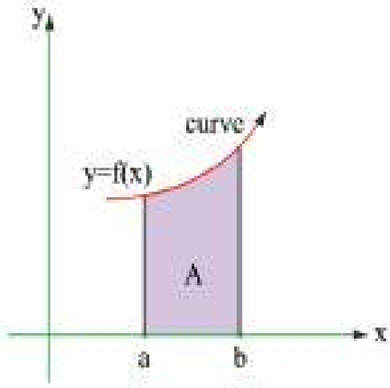
4 - 8 Plane Area by Definite Integral

4 - 8 -1 The Area Between The x-axis and the Curve

Let y = f(x) is continous on [a, b] and A is the area under curve and bounded by the lines x = a, x = b. and x−axis

If f(x) > 0 then area A is $A = \int_{a}^{b} f(x) \, dx$:

If f(x) < 0 then area A is $A = -\int_{a}^{b} f(x) \, dx$:



(Figure 4-23)

04 INTEGRATION

In order to calculate the area under the curve and bounded by x–axis and the lines x = a, x = b.

1. Find the points where f(x) = 0
2. We use the points which make f(x) = 0 and we determine the partitions. on [a , b]
3. We calculate the each integral for the partitions.
4. We add the absolute value of each integral found in (3)

Example 1

Find the area of the region under the curve $f(x) = x^3 - 4x$ and bounded by x–axis on the interval [−2, 2]

Solution

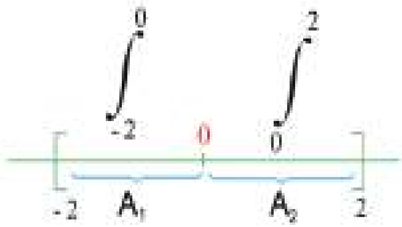
First step :

$f(x) = 0$

$\therefore x^3 - 4x = 0$

$x(x^2 - 4) = 0$

$x(x-2)(x+2) = 0$



$\therefore x = 0, x = 2, x = -2$

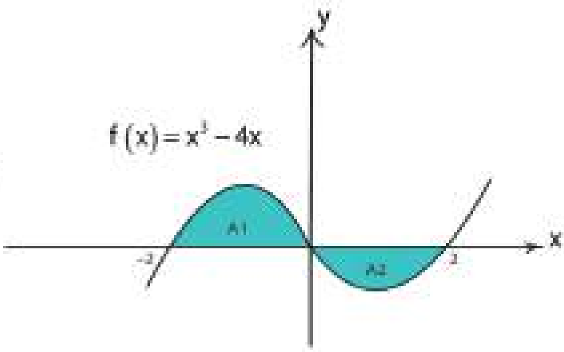
Second step :

The intervals are [-2, 0], [0, 2]

Third step:

$$A_1 = \int_{-2}^{0} (x^3 - 4x)\,dx = \left[\frac{x^4}{4} - 2x^2\right]_{-2}^{0} = 0 - \left[4 - 8\right] = 4$$

$$A_2 = \int_{0}^{2} (x^3 - 4x)\,dx = \left[\frac{x^4}{4} - 2x^2\right]_{0}^{2} = \left[4 - 8\right] - 0 = -4$$



Fourth step:

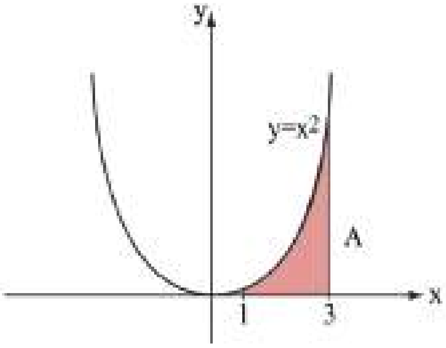
We add the absolute value of the result

$A = |A_1| + |A_2| \Rightarrow A = |4| + |-4| = 4 + 4 = 8$ unit^2

Example 2

Find the area of the region under the curve $y = x^2$ and bounded by x-axis and lines $x = 1$, $x = 3$.

Solution



(Figure 4-24)

$y = 0 \Rightarrow x^2 = 0$

$x = 0 \notin [1, 3]$

$f(x) \geq 0,\ x \in [1, 3]$

$$A = \int_1^3 x^2 dx = \left[\frac{x^3}{3}\right]_1^3 = \frac{27}{3} - \frac{1}{3} = \frac{26}{3} = 8\frac{2}{3} \text{ unit}^2$$

Example 3

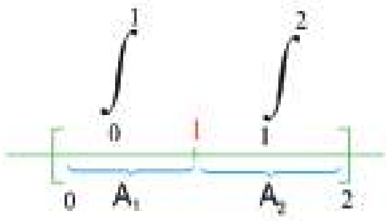
Find the area of the region bounded by the curve. $y=f(x)=x^3-3x^2+2x$ and the x-axis

Solution

$y = 0$

$x^3 - 3x^2 + 2x = 0 \Rightarrow x(x-1)(x-2) = 0$

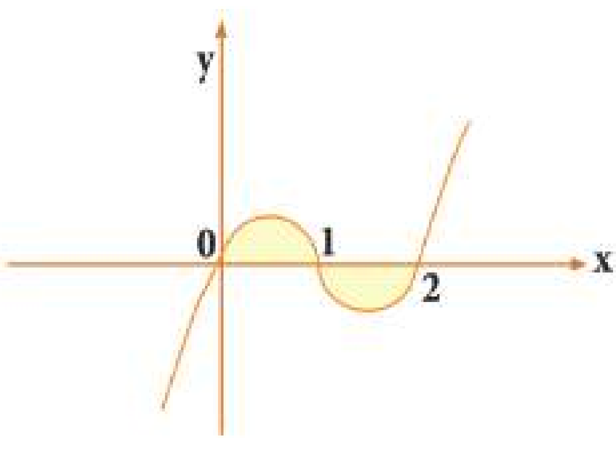
$x = 0,\ x = 1,\ x = 2$



the intervals are [0, 1], [1, 2]

$$A_1 = \int_0^1 (x^3 - 3x^2 + 2x)dx = \left[\frac{x^4}{4} - x^3 + x^2\right]_0^1$$

$$A_2 = \int_1^2 (x^3 - 3x^2 + 2x)dx = \left[\frac{x^4}{4} - x^3 + x^2\right]_1^2$$



(Figure 4-25)

$A_1 = \left(\frac{1}{4} - 1 + 1\right) - (0) = \frac{1}{4}$ $\qquad$ $A_2 = (4 - 8 + 4) - \left(\frac{1}{4} - 1 + 1\right) = -\frac{1}{4}$

$A = |A_1| + |A_2|$

$\therefore A = \left|\frac{1}{4}\right| + \left|\frac{-1}{4}\right| = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ unit^2

Example 4

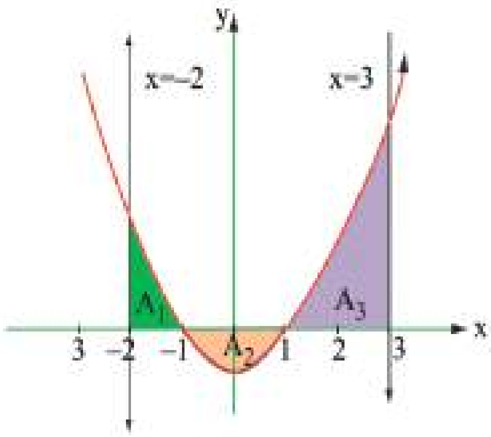
Find the area of the region bounded by the curve $y = x^2 - 1$ and the x–axis on the interval $[-2, 3]$

Solution

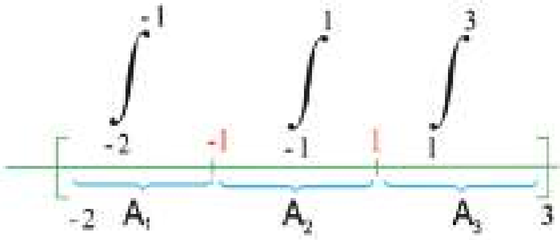
x–intercepts,

$y = 0 \Rightarrow 0 = x^2 - 1 \Rightarrow x = \pm 1 \in [-2, 3]$

the intervals are $[-2, -1]$, $[-1, 1]$, $[1, 3]$



(Figure 4-26)



$$A_1 = \int_{-2}^{-1} (x^2 - 1)\,dx = \left[\frac{x^3}{3} - x\right]_{-2}^{-1} = \left[\frac{-1}{3} + 1\right] - \left[\frac{-8}{3} + 2\right] = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$$

$$A_2 = \int_{-1}^{1} (x^2 - 1)\,dx = \left[\frac{x^3}{x} - x\right]_{-1}^{1} = \left[\frac{1}{3} - 1\right] - \left[\frac{-1}{3} + 1\right] = -\frac{2}{3} - \frac{2}{3} = -\frac{4}{3}$$

$$A_3 = \int_{1}^{3} (x^2 - 1)\,dx = \left[\frac{x^3}{3} - x\right]_{1}^{3} = [9 - 3] - \left[\frac{1}{3} - 1\right] = 6 + \frac{2}{3} = \frac{20}{3}$$

$$A = |A_1| + |A_2| + |A_3|$$

$$\therefore \quad A = \left|\frac{4}{3}\right| + \left|-\frac{4}{3}\right| + \left|\frac{20}{3}\right| = 9\frac{1}{3} \qquad \text{unit}^2$$

Example 5

Find the area of the region bounded by the curve $y = \sin x$ and the x–axis on the interval

$\left[-\frac{\pi}{2}, \pi\right]$.

Solution

$\because \ \sin x = 0 \Rightarrow x = 0 + n\pi \ , \ n \in Z$

$$\therefore \ n = 0 \Rightarrow x = 0 \in \left[-\frac{\pi}{2}, \pi\right]$$

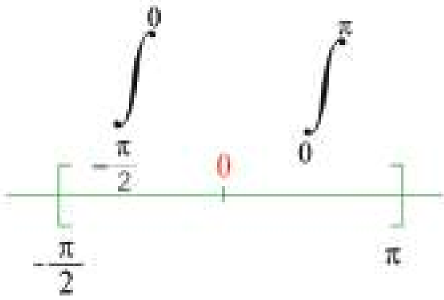
$$n = 1 \Rightarrow x = \pi \in \left[-\frac{\pi}{2}, \pi\right]$$

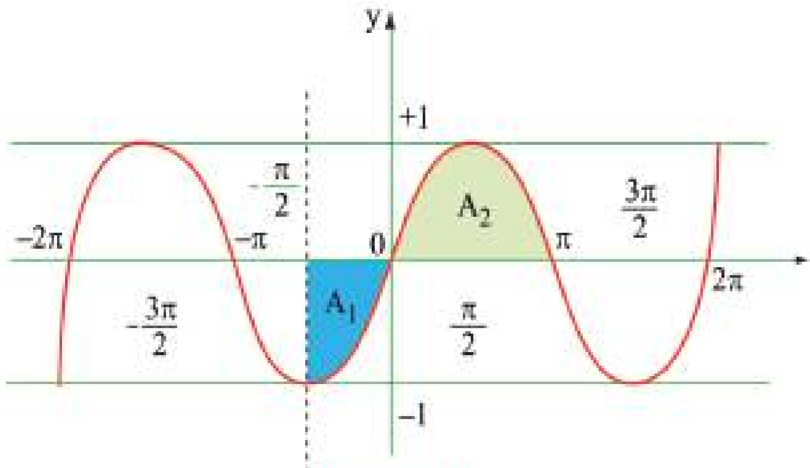
$$n = 2 \Rightarrow x = 2\pi \notin \left[-\frac{\pi}{2}, \pi\right]$$

$$n = -1 \Rightarrow x = -\pi \notin \left[-\frac{\pi}{2}, \pi\right]$$

$$n = -2 \Rightarrow x = -2\pi \notin \left[-\frac{\pi}{2}, \pi\right]$$

$\therefore$ the intervals are $\left[-\frac{\pi}{2}, 0\right]$ and $[0, \pi]$





(Figure 4-27)

$$A_1 = \int_{-\frac{\pi}{2}}^{0} \sin x\,dx = \left[-\cos x\right]_{-\frac{\pi}{2}}^{0} = -\cos(0) + \cos\left(-\frac{\pi}{2}\right)$$

$$A_1 = -1 + 0 = -1$$

$$A_2 = \int_{0}^{\pi} \sin x\,dx = \left[-\cos x\right]_{0}^{\pi} = -\cos\pi + \cos 0$$

$$A_2 = 1 + 1 = 2$$

$$A = |A_1| + |A_2|$$

$$\therefore A = |-1| + |2| \Rightarrow A = 3 \quad \text{unit}^2$$

Example 6

Find the area of the region bounded by the curve y =cos x and the x–axis on the interval [−π , π]

Solution

$$y = 0 \Rightarrow \cos x = 0 \Rightarrow x = \frac{\pi}{2} + n\pi,\ n \in Z$$

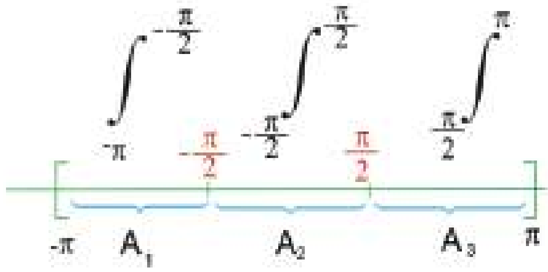
$$\therefore \quad n = 0 \Rightarrow x = \frac{\pi}{2} \in [-\pi, \pi]$$

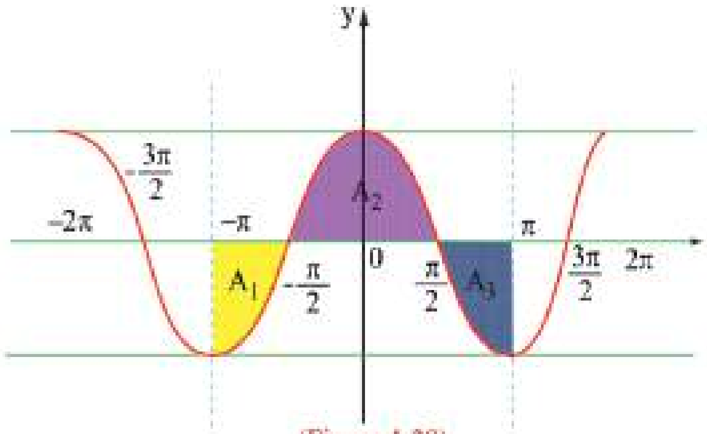
$$n = -1 \Rightarrow x = -\frac{\pi}{2} \in [-\pi, \pi]$$

$$n = 1 \Rightarrow x = \frac{3\pi}{2} \notin [-\pi, \pi]$$

$$n = -2 \Rightarrow x = -\frac{3\pi}{2} \notin [-\pi, \pi]$$

The intervals are $\left[-\pi, -\frac{\pi}{2}\right], \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \left[\frac{\pi}{2}, \pi\right]$





(Figure 4-28)

$$A_1 = \int_{-\pi}^{-\frac{\pi}{2}} \cos x \, dx = \left[\sin x\right]_{-\pi}^{-\frac{\pi}{2}} \qquad A_1 = \sin\left(-\frac{\pi}{2}\right) - \sin(-\pi) = -\sin\frac{\pi}{2} + \sin\pi = -1 + 0 = -1$$

$$A_2 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \, dx = \left[\sin x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right) = 1 + 1 = 2$$

$$A_3 = \int_{\frac{\pi}{2}}^{\pi} \cos x dx = \left[\sin x\right]_{\frac{\pi}{2}}^{\pi} = \sin\pi - \sin\frac{\pi}{2} = 0 - 1 = -1$$

$$A = |A_1| + |A_2| + |A_3|$$

$$A = |-1| + |2| + |-1| = 1 + 2 + 1 = 4 \text{ unit}^2$$

4 - 8 -2 Area Between Two Curves

In order to calculate the area of region between the curves f(x) and g(x) which are continuous on [a, b] follow the steps.

1. If f(x) > g(x) on [a, b] $\quad A = \int_a^b \left[f(x) - g(x)\right] dx.$

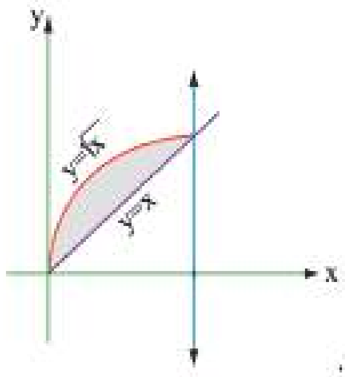
2. If f(x) < g(x) on [a, b] $\quad A = -\int_a^b \left[f(x) - g(x)\right] dx$

3. If two curves are intersected between [a, b] we find the intersection points by solving f(x) = g(x).
   After we find the values of x which belongs to (a, b) and then we divide the interval into partitions and we calculate the integral of difference of two curves on each interval, finally, we add the absolute values of each integral.

Example 1

Find the area of the region between $f(x) = \sqrt{x}$ and $g(x) = x$.

Solution



(Figure 4-29)

Let : $\sqrt{x} = x$

$\therefore x = x^2 \Rightarrow x(x-1) = 0$

$\therefore x = 0,\ x = 1 \Rightarrow x \in [0, 1]$

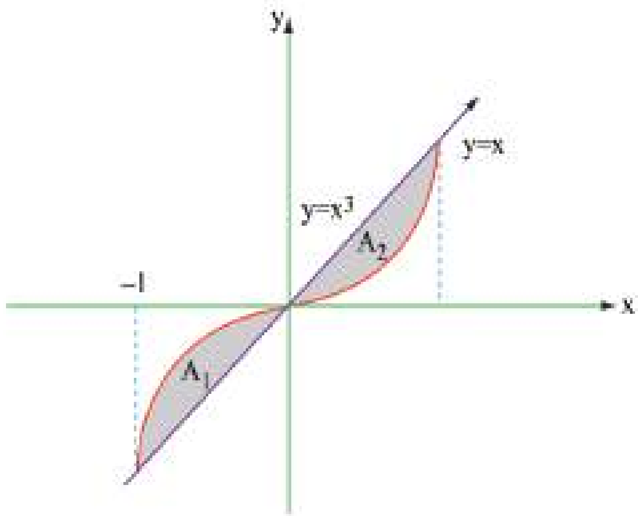
$$A = \int_0^1 \left(\sqrt{x} - x\right)dx = \left[\frac{2}{3}\sqrt{x^3} - \frac{x^2}{2}\right]_0^1 = \left[\frac{2}{3} - \frac{1}{2}\right] - [0] = \frac{1}{6}$$

$$\therefore A = \left|\frac{1}{6}\right| = \frac{1}{6} \text{ unit}^2$$

Example 2

Find the area of the region between curves $y = x^3$ and $y = x$.

Solution

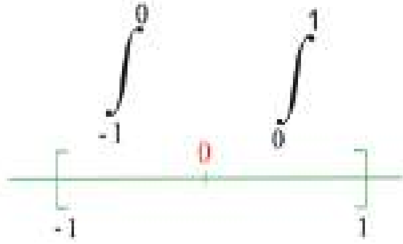


(Figure 4-30)

Let $x^3 = x$

$x^3 - x = 0 \Rightarrow x(x-1)(x+1) = 0$

$x = 0, x = -1 \Rightarrow [-1, 0], [0, 1]$



$$A = |A_1| + |A_2| = \left| \int_{-1}^{0} (x^3 - x)\,dx \right| + \left| \int_{0}^{1} (x^3 - x)\,dx \right|$$

$$\left| \left[\frac{x^4}{4} - \frac{x^2}{2} \right]_{-1}^{0} \right| + \left| \left[\frac{x^4}{4} - \frac{x^2}{2} \right]_{0}^{1} \right|$$

$$\left| 0 - \left(\frac{1}{4} - \frac{1}{2} \right) \right| + \left| \left(\frac{1}{4} - \frac{1}{2} \right) - 0 \right| = \left| \frac{1}{4} \right| + \left| -\frac{1}{4} \right| = \frac{1}{2} \text{ unit}^2$$

Example 3

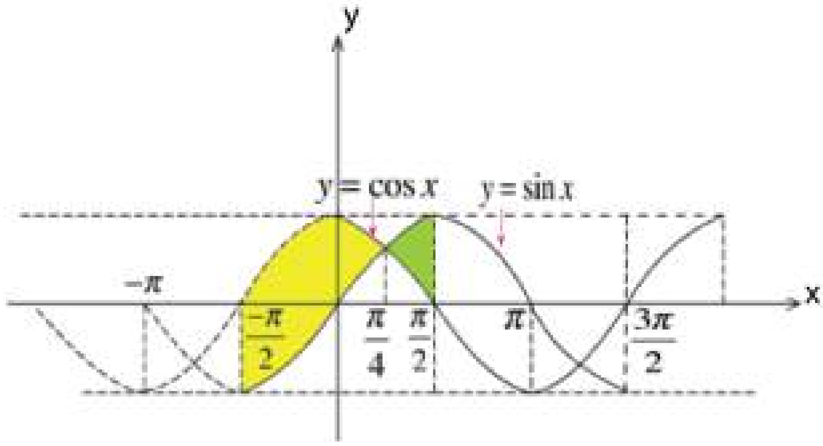
Find the area of the region between curves f(x) = cos x, g(x) = sin x on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$

Solution

$\sin x = \cos x \Rightarrow \tan x = 1$

$$x = \begin{cases} \frac{\pi}{4} \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \\ \frac{5\pi}{4} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \end{cases}$$

$$\left[-\frac{\pi}{2}, \frac{\pi}{4} \right], \left[\frac{\pi}{4}, \frac{\pi}{2} \right]$$



$$A = \left| \int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} (\cos x - \sin x)dx \right| + \left| \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cos x - \sin x)dx \right|$$

$$= \left| \left[\sin x + \cos x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \right| + \left| \left[\sin x + \cos x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \right|$$

$$\left| \left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) - \left(\sin\frac{-\pi}{2} + \cos\frac{-\pi}{2}\right) \right| + \left| \left(\sin\frac{\pi}{2} + \cos\frac{\pi}{2}\right) - \left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) \right|$$

$$A = |\sqrt{2} + 1| + |1 - \sqrt{2}| = \sqrt{2} + 1 + \sqrt{2} - 1 = 2\sqrt{2} \text{ unit}^2$$

4 - 8 - 3 The Distance

Let V(t) be the velocity of an object moving on the straight line the distance which is traveled on the time interval $[t_1, t_2]$ is

$$d = \int_{t_1}^{t_2} |v(t)| dt$$

$$\text{displacement} = S = \int_{t_1}^{t_2} v(t)\, dt$$

S is displacment

$$\text{velocity} = v(t) = \int a(t)dt$$

(a is acceleration)

04 INTEGRATION

Example 1

An object is moving on a straight line at a velocity

$v(t) = 2t - 4$ m/s. Find

a) The distance traveled on [1, 3]

b) Displasement on [1, 3]

c) The distance traveled in fifth second.

d) Dislacement after 4 seconds from initial motion.

Solution

a) $\therefore 2t - 4 = 0 \Rightarrow t = 2 \in [1,3] \Rightarrow [1,2],[2,3]$

$$d = \left|\int_1^2 (2t-4)dt\right| + \left|\int_2^3 (2t-4)dt\right| = \left|\left[t^2 - 4t\right]_1^2\right| + \left|\left[t^2 - 4t\right]_2^3\right|$$

$$= \left|(4-8)-(1-4)\right| + \left|(9-12)-(4-8)\right| = 1 + 1 = 2m$$

b) $s = \int_1^3 (2t-4)dt = \left[t^2 - 4t\right]_1^3 = \left[9-12\right]-\left[1-4\right] = 0$

c) $d = \left|\int_4^5 (2t-4)dt\right| = \left|\left[t^2 - 4t\right]_4^5\right| = \left|\left[25-20\right]-\left[16-16\right]\right| = 5\text{ m}$

d) $s = \int_0^4 (2t-4)dt = \left[t^2 - 4t\right]_0^4 = \left[16-16\right]-\left[0\right] = 0$

Example 2

An object is moving on a straight line an acceleration of 18 m/s^2, if its velocity 82 m/s after 4 seconds from initial motion,Find

a) Distance during the third second

b) Its displacement from initial motion point after 3 seconds.

Solution

a) $v = \int a(t)dt \Rightarrow v = \int 18\,dt$

$v = 18t + c \qquad v = 82,\ t = 4$

$82 = (18)(4) + c \Rightarrow c = 10$

$v = 18t + 10$

$18t+10 > 0 \Rightarrow v > 0$

$$d = \int_2^3 (18t + 10)dt = \left[9t^2 + 10t\right]_2^3 = [81 + 30] - [36 + 20] = 55\ m$$

b) $$s = \int_0^3 (18t + 10)dt = \left[9t^2 + 10t\right]_0^3 = [81 + 30] - [0] = 111m$$

04 INTEGRATION

Exercises

1) Find the area of region bounded by the curve $y=x^4-x$,x-axis and the lines $x=-1$, $x=1$.

2) Find the area of region bounded by the function $f(x)=x^4-3x^2-4$ and x-axis on

the interval $[-2,3]$

3) Find the area of region bounded by the function $f(x)=x^4-x^2$ and x - axis.

4) Find the area of region bounded by the curve $y=\sin 3x$ and x-axis on the interval $\left[0,\frac{\pi}{2}\right]$

5) Find the area of region bounded by the curve $y=2\cos^2 x-1$ and x-axis on the interval $\left[0,\frac{\pi}{2}\right]$

6) Find the area of region bounded by the two functions $y=\frac{1}{2}x$ and $y=\sqrt{x-1}$ on the interval $[2,5]$

7) Find the area of region bounded by the two functions $y=x^2$, $y=x^4-12$

8) Find the area of region bounded by the two functions $f(x)=\sin x$ and $g(x)=\sin x\cos x$ such that $x\in[0,2\pi]$

9) Find the area of region bounded by the two functions $f(x)=2\sin x+1$,$g(x)=\sin x$ such that $x\in\left[0,\frac{3\pi}{2}\right]$

10) Find the area of region bounded by the two functions $y=x^3+4x^2+3x$ and x-axis

11) An object is moving on a straight line with the velocity $v(t)=3t^2-6t+3$ m/s calculate

a) Distance traveled during the interval $[2,4]$

b) Displacement during the interval [0,5]

12) An object is moving on a straight line with acceleration $a(t)=4t+12\ m/s^2$. If the velocity after 4 seconds from initial motion is 90 m/s, then calculate

a) Velocity at t=2

b) Distance traveled during the interval [1,2]

c) The displacement after 10 seconds from initial motion .

13) A point is moving from stillness and after t seconds it's velocity becomes $100t-6t^2$ m/s find the required time to return it to it's initial position. Then calculate the acceleration at that time.

4 - 9 Volumes of Revolution

1. The volume a shape which is formed by rotating the region bounded by the continuous function $y = f(x)$ from $x = a$ to $x = b$ about the x–axis.

$$v = \pi \int_a^b y^2 . dx$$

2. The volume a shape which is formed by rotating the region bounded of by the continuous function $x = f(y)$ from $y = a$ to $y = b$ about the y–axis.

$$v = \pi \int_a^b [x]^2 dy$$

Example 1

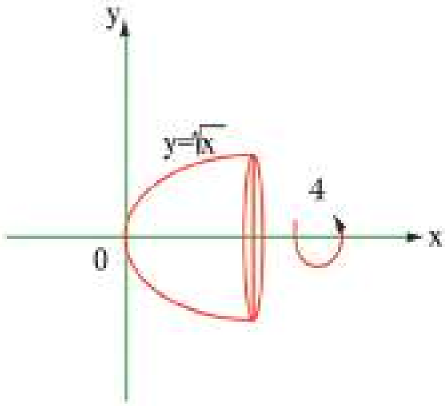
Find the volume of a solid shape which is formed by rotating $y = \sqrt{x}$ on [0, 4] about x–axis.

Solution

$$V = \int_a^b \pi y^2 dx$$

$$= \int_0^4 \pi(\sqrt{x})^2 dx = \int_0^4 \pi x \, dx$$

$$= \left[\pi \frac{x^2}{2}\right]_0^4 = 8\pi - 0 = 8\pi \text{ unit}^3$$



(Figure 4-31)

Example 2

Find the volume of $x = \frac{1}{\sqrt{y}}$ after rotating on $1 \le y \le 4$ about the y–axis.

Solution

$$V = \int_1^4 \pi x^2 dy = \int_1^4 \pi . \frac{1}{y} dy = \left[\pi \ln y\right]_1^4 = \pi \ln 4 - 0 = 2\pi \ln 2 \text{ unit}^3$$

Example 3

Find the volume of shape formed by rotating the region of parabola whose equation is $y^2 = 8x$ and the lines $x = 0$, $x = 2$ about the x–axis.

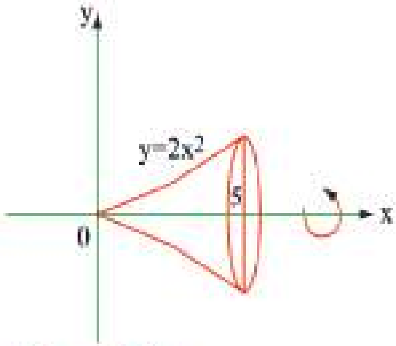
Solution

$$V=\pi\int_a^b y^2dx=\pi\int_0^2 8x\,dx=4\pi\left| x^2\right|_0^2=16\pi \text{ unit}^3$$

Example 4

Find the volume of shape which is formed by rotating the region of parabola whose equation is $y=2x^2$ and the lines $x=0$, $x=5$ about the x-axis.

Solution



(Figure 4-32)

$$V=\pi\int_a^b y^2dx=\pi\int_0^5 4x^4dx=\frac{4\pi}{5}\left[x^5\right]_0^5$$

$$=\frac{4\pi}{5}\cdot 3125=2500\pi \text{ unit}^3$$

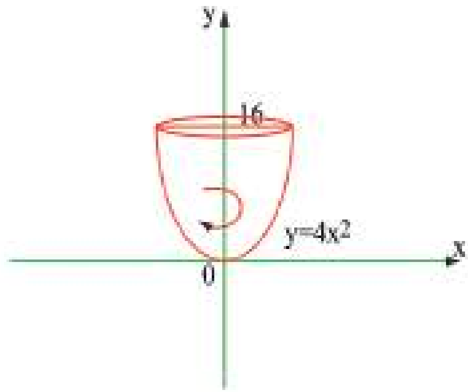
Example 5

Find the volume of shape which is formed by rotating the region of parabola whose equation is $y=4x^2$ and the lines $y=0$, $y=16$ about the y-axis.

Solution

$$V=\pi\int_a^b x^2dy$$

$$V=\pi\int_0^{16}\frac{y}{4}dy=\frac{\pi}{8}\left[y^2\right]_0^{16}=\frac{\pi}{8}\left[16\,(16)\right]=32\pi \text{ unit}^3$$



Example 6

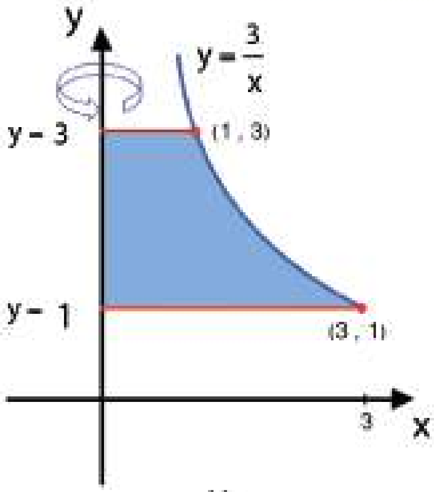
Find the volume of shape which formed by rotating the region of the curve $y=\frac{3}{x}$ and the lines $1\le y\le 3$ about the y-axis

Solution

$y = 1 \Rightarrow x = 3$

$y = 3 \Rightarrow x = 1$

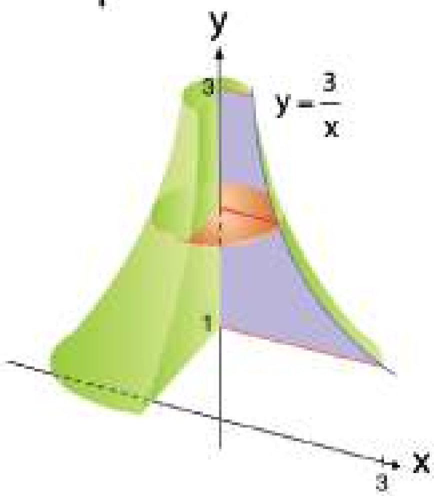
$$V = \pi \int_a^b x^2 \, dy$$



$$V = \pi \int_1^3 \left[\frac{3}{y}\right]_1^2 dy$$

$$= 9\pi \left[\frac{-1}{y}\right]_1^3$$

$$= 9\pi \left[\frac{-1}{3} + 1\right] = 6\pi \text{ unit}^3$$



Exercises

1) Find the volume of the shape which is formed by rotating the parabola whose equation is $y=x^2$ and the lines $x=1$ and $x=2$ about the x - axis.

2) Find the volume of the shape which is formed by rotating the curve $y=x^2+1$ and the line $y=4$ about the y - axis.

3) Find the volume of the shape which is formed by rotating the $y^2+x=1$ and the line $x=0$ about the y - axis.

4) Find the volume of the shape which is formed by rotating the curve whose equation is $y^2=x^3$ and the lines $x=0$ and $x=2$ about the x-axis.

ORDINARY DIFFERENTIAL EQUATIONS 05

Term	Symbol or Mathematical Relation
Ordinary Differential Equation	O.D.E

5-1 Introduction

Ordinary Differential Equations is a major subject in applied Mathematics, it has relevance to many scientific disciplines and engineering. In this chapter, we will discuss the differential equations and ways to solve them.

Definition 5-1-1

Differential equation is the equation, which contains one or more derivatives for unknown function (i.e. the dependant variable in equation).

Note

Ordinary Differential Equation is the relationship between independent variable, let it be (x) and its unknown function (y) (Dependant Variable) and some (y) derivatives according to (x), it is symbolized as O.D.E. which stands for (Ordinary Differential Equation).

05 ORDINARY DIFFERENTIAL EQUATIONS

For Example

1. $\frac{dy}{dx} = 3y - 4x$
2. $x^2y'' + 5xy' - x^3y = 0$
3. $\frac{d^3y}{dx^3} + \frac{dy}{dx} = y - 4$
4. $y' + x^2y + x = y$
5. $(y'')^3 + 2y' + x^2 \ln x = 5$
6. $y^{(4)} + \cos y + x^2\, y\, y' = 0$

All of the above are Ordinary Differential Equations because y variable depends only on x variable.

Definition 5-1-2

Order: Order of Differential Equation is defined as order of highest derivative.

Degree: Degree of Differential Equation is defined as highest exponent of the highest derivative in Differential Equation.

For Example

1. $\frac{dy}{dx} + x - 7y = 0$	first order, first degree.
2. $\frac{d^2y}{dx^2} = 5x - 3xy + 7$	second order, first degree.
3. $(y''')^3 + y' - y = 0$	third order, third degree.
4. $y'' + 2y(y')^3 = 0$	second order, first degree.
5. $\left(\frac{dy}{dx}\right)^4 = x^3 - 5$	first order, fourth degree.
6. $x^2\left(\frac{dy}{dx}\right)^4 + \left(\frac{d^3y}{dx^3}\right)^2 + 2\frac{d^2y}{dx^2} = 0$	third order, second degree.
7. $y^{(4)} + \cos y + x^2y\, y' = 0$	fourth order, first degree.

Note

Degree of differential equation with algebraic derivatives is the algebraic degree of the derivative with highest order in the equation. For example, the differential equation: $(y'')^2 = \sqrt{1+(y')^2}$ is the the second order, because y'' is the highest derivative in this equation.

Whereby, roots and fractal exponents can be removed, we get: $(y'')^4 = 1 + (y')^2$ Thus, the differential equation is in the fourth Degree.

5-2 Solution of an Ordinary Differential Equation

The aim of studying differential equation is to find solution for them, this is done by finding a relation between the dependant variable Y and independent variable X, in such a way that the relation shipis free or derivatives (without derivatives) and the differential equation is satisfied by substitution.

Definition 5-2-1

Solution of differential equation is the relatonship between its variables, in such a way that:

a. Derivaties free (without derivatives).

b. Defined on certain interval

c. Satisfy the differential equation

i.e solution of ordinary differential equation is any function of an (unknown) (dependent variable) indicated by indepentent variable which satisfies the differential equation.

05 ORDINARY DIFFERENTIAL EQUATIONS

Example 1

Show that the relation $y = x^2 + 3x$ is a solution of differential equation $xy' = x^2 + y$

Solution

$y = x^2 + 3x$, we find y', thus:

$y = x^2 + 3x$ ... ① $\Rightarrow y' = 2x + 3$ ...②

Substitute ① and ② in the right hand-side and left hand-side of the differential equation, as follows:

$LHS = xy'$

$= x(2x + 3) = 2x^2 + 3x$

$RHS = x^2 + y = x^2 + x^2 + 3x$

$= 2x^2 + 3x = LHS$

Therefore, the given relation is a solution of the differential equation above.

5-3 particular and General Solution of Ordinary Differential Equation

The solution of the ordinary differential equation is any relation between x,y and satisfies the equation. However, the general solution of any differential equation is the one, which includes a number of arbitrary constants equal to equation order. If the equation is in first order, its general solution must include one arbitrary constant i.e. integral constant, which emerges in doing integral step for first order equation. However, if the equation is second order, its solution must include two integral constants because we have two integral steps when solving second order equation and so on...

For Example

$$\frac{dy}{dx} - 5y = 0$$

It is a first order differential equation and satisfied by particular solution $y = e^{5x}$, as it appears in substitution of differential equation, yet, the general solution must include one arbitrary constant c, thus: $y = ce^{5x}$

As for differential equation, $\frac{d^2y}{dx^2} + y = 0$ it is second order and satisfied by general solutions: y= sinx, y=cosx, although the general solution must include two arbitrary integral constants, like A, B, hence, the general solution is $y = A \sin x + B \cos x$.

Example 2

Prove that $y = x \ln|x| - x$ is a solution equation $x\frac{dy}{dx} = x + y$, $x > 0$...(1).

Solution

The equation $y = x \ln|x| - x$ is without derivatives and defined on $x > 0$, to prove that it is one solution of differential equation (1), we directly substitute in (1)

$$LHS = x\frac{dy}{dx} = x\left(x.\frac{1}{x} + \ln|x|(1) - 1\right)$$
$$= x.\left(\cancel{1} + \ln|x| - \cancel{1}\right) = x\ \ n|x|$$
$$RHS = x + y = x + x.\ n|x| - x = x.\ n|x|$$

Thus, the given relation is one of the special solutions of differential equation (1)

Example 3

Show that $a \in R$, $\ln y^2 = x + a$ is a solution for the equation $2y' - y = 0$

Solution

$\ln y^2 = x + a \Rightarrow 2 \ln|y| = x + a \Rightarrow 2\frac{1}{y}(y') = 1$

$\Rightarrow 2y' = y \Rightarrow 2y' - y = 0$

$\therefore \ln y^2 = x + a$ is a solution of the above equation.

Example 4

Is $y = x^3 + x - 2$ a solution of the differential equation $\frac{d^2y}{dx^2} = 6x$?

Solution

$$y = x^3 + x - 2 \Rightarrow \frac{dy}{dx} = 3x^2 + 1 \Rightarrow \frac{d^2y}{dx^2} = 6x$$

Thus $y = x^3 + x - 2$ is a solution of the equation $\frac{d^2y}{dx^2} = 6x$

Example 5

Prove that $y = 3\cos 2x + 2\sin 2x$ is a solution of the differential equation $y'' + 4y = 0$.

Solution

$\because y = 3\cos 2x + 2\sin 2x$ ... ①

$\therefore y' = -6\sin 2x + 4\cos 2x$

$\because y'' = -12\cos 2x - 8\sin 2x$ ... ②

Substitute ①, ② in left hand - side of differential equation, we get:

LHS $= (-12\cos 2x - 8\sin 2x) + 4(3\cos 2x + 2\sin 2x) \Rightarrow$

$-12\cos 2x - 8\sin 2x + 12\cos 2x + 8\sin 2x = 0$ Right hand - side

= RHS

Thus, $y = 3\cos 2x + 2\sin 2x$ is a solution of the above equation

Example 6

Is $y^2 = 3x^2 + x^3$ a solution of the equation $yy'' + (y')^2 - 3x = 5$?

Solution

$\because \; y^2 = 3x^2 + x^3 \Rightarrow 2yy' = 6x + 3x^2 \Rightarrow$

$2y\,(y'') + y'(2)\,y' = 6 + 6x$

Divided by 2

$yy'' + (y')^2 = 3 + 3x \Rightarrow \text{LHS} = yy'' + (y')^2 - 3x = 3 \neq 5$ Right hand - side $\neq$ RHS

Thus, $y^2 = 3x^2 + x^3$ is not a solution of the above equation.

Example 7

Show that $y = e^{2x} + e^{-3x}$ is a solution of the differential equation $y'' + y' - 6y = 0$.

Solution

$\because \; y = e^{2x} + e^{-3x} \Rightarrow y' = 2e^{2x} - 3e^{-3x} \Rightarrow y'' = 4e^{2x} + 9e^{-3x}$

Substitute left-hand side of equation

$\text{LHS} = y'' + y' - 6y$

$= (4e^{2x} + 9e^{-3x}) + (2e^{2x} - 3e^{-3x}) - 6(e^{2x} + e^{-3x})$

$= 4e^{2x} + 9e^{-3x} + 2e^{2x} - 3e^{-3x} - 6e^{2x} - 6e^{-3x}$

$= 0 =$ Right hand-side

LHS= RHS

Thus, $y = e^{2x} + e^{-3x}$ is a solution of the above equation.

05 ORDINARY DIFFERENTIAL EQUATIONS

Exercises

1. Show order and degree of the following differential equations :

a) $\left(x^2-y^2\right)+3xy\frac{dy}{dx}=0$

b) $\frac{d^2y}{dx^2}+x\frac{dy}{dx}-5y=7$

c) $(y''')^3-2y'+8y=x^3+\cos x$

d) $\left(\frac{d^3y}{dx^3}\right)^2-2\left(\frac{dy}{dx}\right)^5+3y=0$

2. Prove that $y = \sin x$ is a solution of the equation $y'' + y = 0$

3. Prove that the relation $s = 8\cos 3t + 6\sin 3t$ is a solution of the equation $\frac{d^2s}{dt^2}+9s=0$

4. Is $y = x + 2$ a solution of the equation $y'' + 3y' + y = x$?

5. Is $y = \tan x$ a solution of the equation $y'' = 2y(1 + y^2)$?

6. Is $2x^2 + y^2 = 1$ a solution of the equation $y^3y'' = -2$?

7. Is $yx = \sin 5x$ is solution of the equation $xy'' + 2y' + 25yx = 0$?

8. Show that $y = ae^{-x}$ is a solution of equation $y' + y = 0$ where $a \in R$

9. Show that $c \in R$, $\ln|y| = x^2 + c$ is a solution of the equation $y'' = 4x^2y + 2y$?

5-4 1st Order 1st Degree Ordinary Differential Equation

Introduction

Solving differential equation is opposite to differentiation operation ,i.e. it depends on integral operations. Usually, finding the opposite differentiation (direct)of each function is not possible. We do not expect to have a general solution for each differential equation depending on initial common functions. Therefore, solvable differential equations can be divided into several types according to ways of finding their general solution. In this chapter, we will discuss first order

and first degree differential equations with two variables x,y. This type of equations is simple, although finding a general solution for them is not possible, and there is no general way of solvin them. Therefore, these equations will be divided according to way of solving them:

1. Separation of variables equations
2. Homogenous differential equations
3. Exact differential equations
4. Linear differential equation- Bernoulli

In this chapter, we will discuss (1) only .

For Example, First order first-degree differential equation has following

two forms:

1. $\frac{dy}{dx} = F(x, y)$
2. $M(x, y)\, dx + N(x, y)\, dy = 0$

   Whereby $N(x, y) \neq 0$, $M(x, y) \neq 0$

For Example

The differential equation $\frac{dy}{dx} = \frac{3xy}{x + y}$

Can be rewritten as

$(3xy)dx = (x + y)dy$

$(3xy).dx - (x + y).dy = 0$

Whereby $M = 3xy$, $N = (x + y)$

In the next section, we will study some ways of solving differential equation.

5-5 The methods of Solving Differential Equation

Separation of Variables

In this type of equations, as it appears from the name, we can separate all the (terms) which contain x only with dx on one side, and the terms which contain y only with dy on the other side, so we get:

$$f(x)\,.\,dx = g(y)\,dy \quad ...(1)$$

Then integrate sides of equation (1)

$$\int g(y)\,dy = \int f(x)\,dx + c$$

Whereby, c is arbitrary constant.

Example 1

Solve the equation $\frac{dy}{dx} = 2x + 5$.

Solution

$$\frac{dy}{dx} = 2x + 5 \Rightarrow dy = (2x + 5)\,dx$$

$$\int dy = \int (2x + 5)\,dx \Rightarrow y = x^2 + 5x + c$$

Example 2

Solve the equation $\frac{dy}{dx} = \frac{x-1}{y}$.

Solution

Rewrite the equation in the form g(y) dy = f(x) dx

i.e $y\,dy = (x - 1)\,dx$

Integrate both sides $\int y\,dy = \int (x-1)\,dx$

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 - x + c$$

$$y^2 = x^2 - 2x + 2c \Rightarrow y = \pm\left(x^2 - 2x + 2c\right)^{\frac{1}{2}}$$

$$= \pm\left(x^2 - 2x + c_1\right)^{\frac{1}{2}}$$

Where $c_1 = 2c$ is arbitrary constant

Example 3

Solve the differential equation $dy = \sin x \cos^2 y\,dx$ whereby $y \neq (2n+1)\frac{\pi}{2}$, $\cos y \neq 0$

Solution

Rewrite the equation in the form g(y) dy = f(x) dx

i.e.

$$\frac{1}{\cos^2 y}dy = \sin x\,dx$$

$$\sec^2 y\,dy = \sin x\,dx$$

Considering integral $\Rightarrow \int \sec^2 y\,dy = \int \sin x\,dx$

$\tan y = -\cos x + c$ Whereby c is arbitrary constant

Example 4

Find solution of differential equation $y' - x\sqrt{y} = 0$ where $x = 2, y = 9$

Solution

$$y' - x\sqrt{y} = 0 \Rightarrow \frac{dy}{dx} - xy^{\frac{1}{2}} = 0 \Rightarrow \frac{dy}{dx} = xy^{\frac{1}{2}}$$

$$y^{-\frac{1}{2}}dy = xdx \Rightarrow \int y^{-\frac{1}{2}}dy = \int xdx \Rightarrow 2\sqrt{y} = \frac{1}{2}x^2 + c$$

Substitute $x = 2, y = 9$, we get:

$$2\sqrt{9} = \frac{1}{2}(2)^2 + c \Rightarrow 6 = 2 + c \Rightarrow c = 4$$

$\therefore$ the solution is

$$2\sqrt{y} = \frac{1}{2}x^2 + 4 \Rightarrow y = \left(\frac{1}{4}x^2 + 2\right)^2$$

Example 5

Solve the equation $\frac{dy}{dx} = e^{2x+y}$ where $y = 0$ when $x = 0$

Solution

$$\frac{dy}{dx} = e^{2x}.e^{y} \Rightarrow e^{-y}dy = e^{2x}dx$$

$$-\int e^{-y}(-1)dy = \frac{1}{2}\int e^{2x}(2)\,dx$$

$$-e^{-y} = \frac{1}{2}e^{2x} + c$$

Substitute $y= 0, x = 0$, we get:

$$\Rightarrow -e^{-0} = \frac{1}{2}e^{0} + c \Rightarrow -1 = \frac{1}{2} + c \Rightarrow c = -\frac{3}{2}$$

Thus, the solution is:

$$-e^{-y} = \frac{1}{2}e^{2x} - \frac{3}{2} \Rightarrow e^{-y} = \frac{1}{2}(3 - e^{2x})$$

$$\frac{1}{e^y} = \frac{3 - e^{2x}}{2}$$

$$e^y = \frac{2}{3 - e^{2x}}$$

Taking ln for the two sides, we get:

$$y = \ln\left|\frac{2}{3 - e^{2x}}\right|$$

Example 6

Find general solution for differential equation $(x + 1)\frac{dy}{dx} = 2y$

Solution

$$\frac{dy}{y} = 2\frac{dx}{x+1} \Rightarrow \int \frac{dy}{y} = 2\int \frac{dx}{x+1}$$

$$\ln|y| = \ln(x+1)^2 + c \Rightarrow \quad \ln|y| - \ln(x+1)^2 = c \Rightarrow \quad \frac{y}{(x+1)^2} = c \quad \Rightarrow \quad \frac{|y|}{(x+1)^2} = e^c$$

$$\ln|y| = \ln((x+1)^2 . e^c) \Rightarrow$$

$$|y| = e^c (x+1)^2$$

$$y = \mp c_1 (x+1)^2$$

Whereby $c_1 = e^c$ is arbitrary constant

05 ORDINARY DIFFERENTIAL EQUATIONS

Exercises

1. Solve the following differential equations by separation of variables:

a) $y' \cos^3 x = \sin x$

b) $\frac{dy}{dx} + xy = 3x, \quad x = 1, \; y = 2$

c) $\frac{dy}{dx} = (x+1)(y-1)$

d) $(y^2 + 4y - 1)y' = x^2 - 2x + 3$

e) $yy' = 4\sqrt{(1+y^2)^3}$

f) $e^x dx - y^3 dy = 0$

g) $y' = 2e^x y^3, \; x = 0, \; y = \frac{1}{2}$

2. Find general solution of the following differential equations:

a) $xy \frac{dy}{dx} + y^2 = 1 - y^2$

b) $\sin x \cos y \frac{dy}{dx} + \cos x \sin y = 0$

c) $x\cos^2 y \, dx + \tan y \, dy = 0$

d) $\tan^2 y \, dy = \sin^3 x \, dx$

e) $\frac{dy}{dx} = \cos^2 x \cos^2 y$

f) $\frac{dy}{dx} = \frac{\cos x}{3y^2 + e^y}$

g) $e^{x+2y} + y' = 0$

Chapter 6: Space Geometry

6-1 Introduction

6-2 Dihedral angle and perpendicular planes

6-3 Orthogonal Projection on Plane

6-4 Solids

Term	Symbol or Mathematical Relation
Dihedral angle between (x).(y)	$(x)-\overleftrightarrow{AB}-(y)$
Lateral area	L –A
Total area	T – A
Plane x	(x)

(6 - 1) Introduction

We have already learned that both line and plane are infinite set of points. Every two points designate one and only one line. Every three non - collinear (not on one line) designate one plane only, every four points non-straight designate a space. In brief, line has at least two points, plane has at least three points non-collinear, and space has at least four points not coplanar.

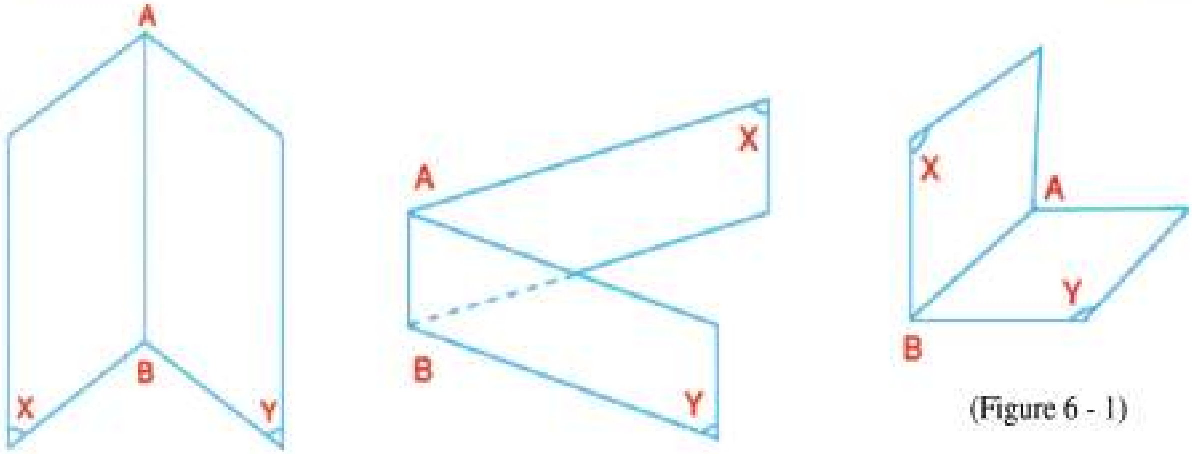
Last year, we learned relations between lines and planes and proved theorems, which can be used in this chapter. Planes are gaining new concepts and new theorems, all you have to do is go back and read this subject from last year.

(6 - 2) Dihedral Angle and Perpendicular Planes

Definition 6-2-1

Dihedral Angle: it is a union of two halves planes with common edge

The common edge is called (Edge of Dihedral), plane halves are called (face of dihedral) as in fig.(6-1)



(Figure 6 - 1)

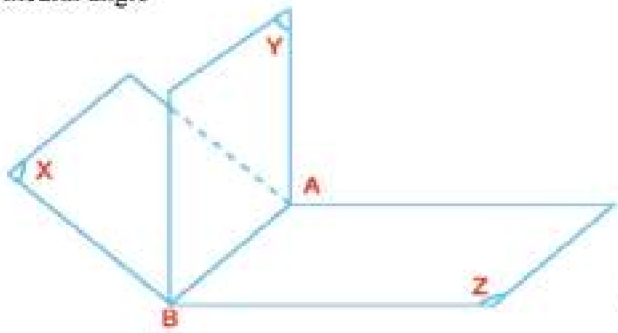
Where $\overleftrightarrow{AB}$ is edge of dihedral angle, (X) ,(Y) are faces of the dihedral angle

Dihedral angle is expressed as (x) - $\overleftrightarrow{AB}$ - (y) ∴ It can also be expressed by edge of dihedral angle if not common with another angle. Example: Dihedral angle

(x) - $\overleftrightarrow{AB}$ - (z)

(x) - $\overleftrightarrow{AB}$ - (y)

(y) – $\overleftrightarrow{AB}$ – (z)

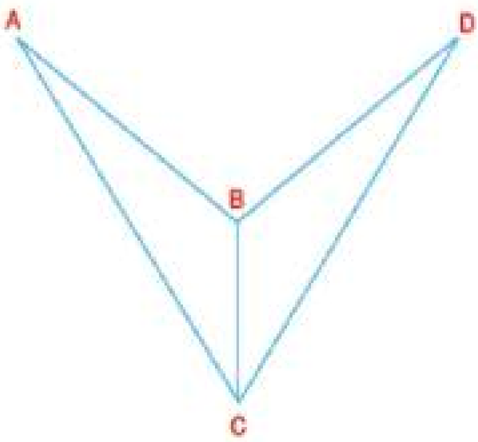


(Figure 6 - 2)

Dihedral angle cannot be written $\overleftrightarrow{AB}$ in this example because the edge $\overleftrightarrow{AB}$ is common with another dihedral angle

Note

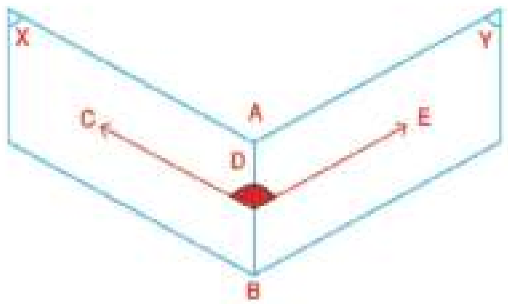
When four points are not coplanar (not in one plane), we write the $A - \overleftrightarrow{BC} - D$ or dihedral angle between planes (ABC), (DBC) (Figure 6 - 3)



(Figure 6 - 3)

Dihedral angle is measured as follows

Designate a point (D) on the common edge $\overleftrightarrow{AB}$, and then draw a perpendicular line $\overrightarrow{DC}$ in (x) from D and perpendicular line $\overleftrightarrow{DE}$ in (Y) on the edge $\overleftrightarrow{AB}$ .The measurement of dihedral angle between the two planes is measurement of angle CDE, it is called CDE angle, which belongs to dihedral angle, as in fig. (6-4)



(Figure 6 - 4)

In other words, we have the dihedral angle $(X)-\overleftrightarrow{AB}-(Y)$ and

$\overrightarrow{DC} \subset (X), \overrightarrow{DE} \subset (Y)$

$\overrightarrow{DC} \perp \overleftrightarrow{AB}, \overrightarrow{DE} \perp \overleftrightarrow{AB}$

$\angle\ CDE$ is the angle which belongs dihedral angle $\overleftrightarrow{AB}$ or $(X)-\overleftrightarrow{AB}-(Y)$

6-2-2 Definition

<u>Half-open plane</u> : Let (M) be a plane and let L $\subset$ (M), the line L divides the plane (M) into three separate sets. (M_1) and (M_2) is called a **half-open plane**, while $(M_1) \cup L$ and $(M_2) \cup L$ each of them is called a **half-plane**. The common line L between two half planes is called the **edge**.

<u>Dihedral Angle</u> : If (M) is a half plane and (N) is a half plane, then they meet in a common line (ab).

So (M) $\cup$ (N) is called a **dihedral angle** and ab is called its edge.

<u>Faces of Dihedral Angle :</u> A **dihedral angle** is a figure formed by two half planes meeting in a common line. The common line is called the edge and half planes are called the **faces of dihedral angle**.

<u>Faces of Dihedral Angle :</u> A **dihedral angle** is a figure formed by two half planes meeting in a common line. The common line is called the edge and half planes are called the **faces of dihedral angle**.

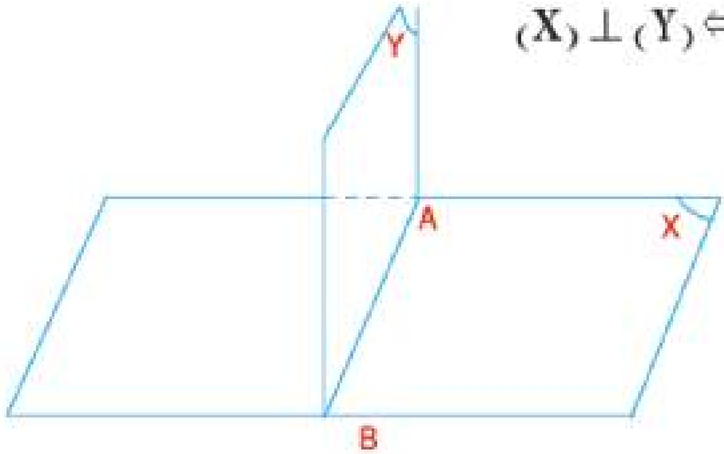
From the definition of both plane angle and dihedral angle, we may conclude:

1) Measurement of plane angle is constant for a dihedral angle.
2) Measurement of dihedral angle equals measurement of plane angle and vice versa.

6-2-3 Definition

If dihedral angle is right angle, then, the planes are perpendicular and vice versa.

$$(X) \perp (Y) \Leftrightarrow (X) - \overleftrightarrow{AB} - (Y) = 90°$$



(Figure 6 - 5)

Theorem 7

"If two planes are perpendicular to each other and if a line which is drawn in one of these planes is perpendicular to the intersection line of two planes, then it is perpendicular to the other plane."

In other words : $(X) \perp (Y)$, $(X) \cap (Y) = \overleftrightarrow{AB}$, $\overleftrightarrow{CD} \subset (Y)$, $\overleftrightarrow{CD} \perp \overleftrightarrow{AB}$ at $D \Rightarrow \overleftrightarrow{CD} \perp (X)$

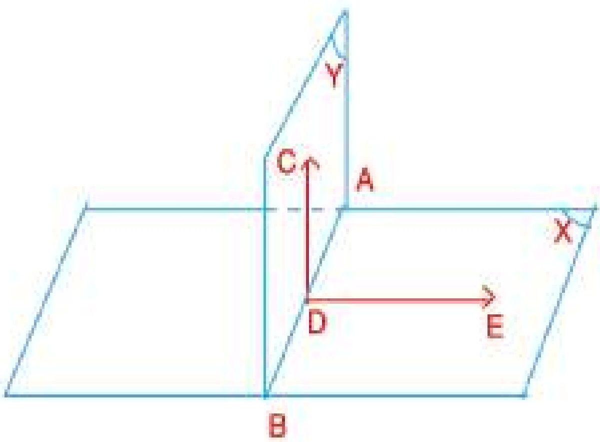
Given :

$(X) \perp (Y)$,

$(X) \cap (Y) = \overleftrightarrow{AB}$

$\overleftrightarrow{CD} \subset (Y)$,

$\overleftrightarrow{CD} \perp \overleftrightarrow{AB}$ at point D.



Required : Prove that $\overleftrightarrow{CD} \perp (X)$

Proof : In (X) draw $\overleftrightarrow{DE} \perp \overleftrightarrow{AB}$

(There is one and only one line perpendicular to a given line in the same plane ....)

$\overleftrightarrow{CD} \subset (Y), \overleftrightarrow{CD} \perp \overleftrightarrow{AB}$ (given)

$\Rightarrow$ $\angle$ CDE is a plane angle of dihedral angle X-$\overleftrightarrow{AB}$-Y. (definition of planel angle)

$(X) \perp (Y)$ (given)

$\Rightarrow$ Measure of the dihedral angle ab $\overleftrightarrow{AB}$ $= 90^0$

(If two planes are perpendicular to each other, then the measurement of the dihedral angle = 90^0)

$\Rightarrow m\angle CDE = 90^0$

(A measure of a dihedral angle is equal to measure of its plane angle and vice versa.

$\Rightarrow \overleftrightarrow{CD} \perp \overleftrightarrow{DE}$

(If a measure of a dihedral angle between two intersecting planes is 90, then the lines are perpendicular and vice versa)

$\Rightarrow \overleftrightarrow{CD} \perp (X)$

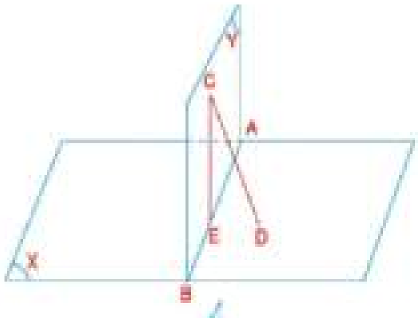
(The line that is perpendicular to two intersecting lines at a point is perpendicular to their plane......)

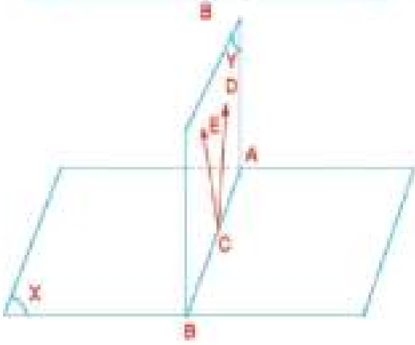
(Q.E.D)

Corollary of Theorem 7

If two planes are perpendicular to each other, and if a line is drawn from any point in one of these planes perpendicular to the other plane, this line must lie in the first plane."

This means:



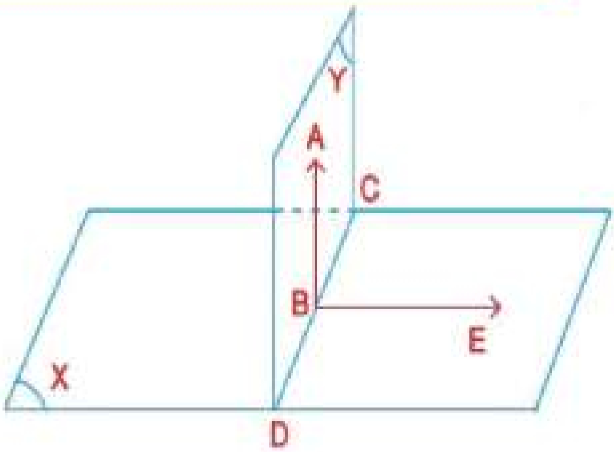


$\overleftrightarrow{CD} \perp (X), C \in (Y), (Y) \perp (X) \Rightarrow \overleftrightarrow{CD} \subset (Y)$

Theorem 8

"If a line is perpendicular to a plane, then any plane containing that line is perpendicular to the given plane."

In other words : Let $\overleftrightarrow{AB} \perp (X)$, $\overleftrightarrow{AB} \subset (Y)$ $\Rightarrow (Y) \perp (X)$



Given:

$\overleftrightarrow{AB} \perp (X)$ at the point B,

$\overleftrightarrow{AB} \subset (Y)$

Required: Prove that $(Y) \perp (X)$

Proof: Let $(X) \cap (Y) = \overleftrightarrow{CD}$ **(If two planes intersect each other, their intersection is a straight lin**

$B \in \overleftrightarrow{CD}$ **(The intersection plane contains the common points)**

In (X) let us draw $\overleftrightarrow{BE} \perp \overleftrightarrow{CD}$.

(At a given point in a given plane one and only one line can be drawn perpendicular to a given line in the plai

Since $\overleftrightarrow{AB} \perp (X)$ (given)

So $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}, \overleftrightarrow{BE}$ **(If a line is perpendicular to a plane, then the line is perpendicular to all lines in the plane and pass through its trace.)**

Since $\overleftrightarrow{AB} \subset (Y)$ (given)

So $\angle ABE$ is plane angle of dihedral angle $\overleftrightarrow{CD}$ **(Definition of plane angle).**

$m\ \angle ABE = 90^0$ (Because $\overleftrightarrow{AB} \perp \overleftrightarrow{BE}$)

$\Rightarrow$ Measure of the dihedral angle $(Y)-\overleftrightarrow{CD}-(X) = 90^0$

(Measure of the dihedral angle is equal to measure of its plane angle and vice versa).

$\Rightarrow (Y) \perp (X)$

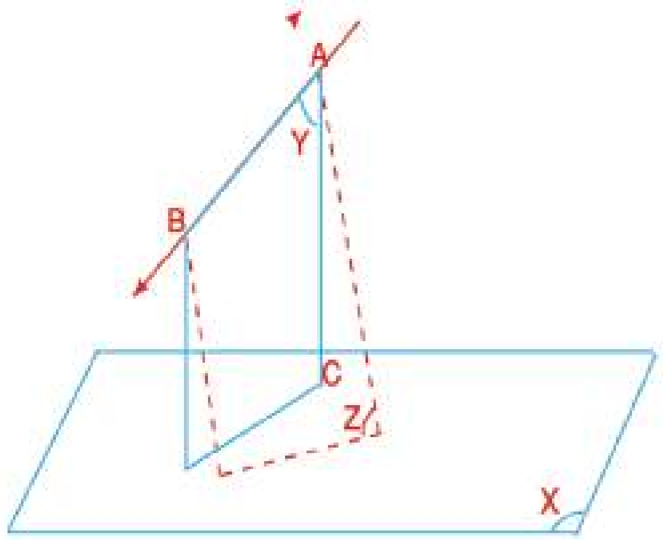
(If the measurement of the dihedral angle = 90^0 , then two planes are perpendicular to each other and vice versa) **(Q.E.D)**

Theorem 9

Through a given external line not perpendicular to a given plane there is one and only one plane perpendicular to the given plane.

This means $\overleftrightarrow{AB}$ is not perpendicular on (X)

There is only one plane contains $\overleftrightarrow{AB}$ and perpendicular on (X)



Given :

$\overleftrightarrow{AB}$ is not perpendicular on (X)

Required: Find a unique plane that contains $\overleftrightarrow{AB}$ and perpendicular on (X)

Proof : At the point (A) ,draw $\overleftrightarrow{AC} \perp (X)$ (there is onunique line perpendicular on a given plane , from a point not belonging to that plane).

$\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$ are Intersect

$\therefore$ There is one unique plane like (Y) containing them (for each intersect lines, there is one plane containing them).

$\therefore$ $(Y) \perp (X)$ (Theorem 8)

To prove Uniqueness:

Let (Z) another plane contains $\overleftrightarrow{AB}$ and perpendicular on (X)

$\because$ $\overleftrightarrow{AC} \perp (X)$ (by proof)

$\therefore$ $\overleftrightarrow{AC} \subset (Z)$ (Corollary of theorem 7)

$\therefore$ $(Y) = (Z)$ (For every two intersect lines, there is one plane contain them)

Q.E.D

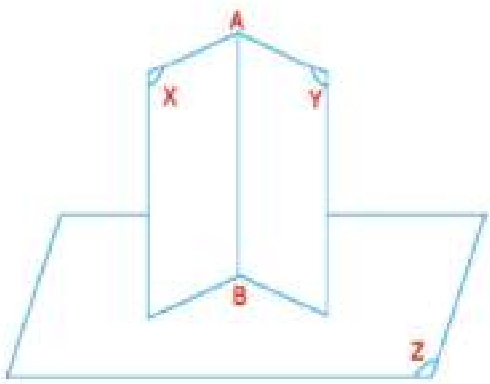
Corollary of Theorem 9

"If two intersecting planes are each perpendicular to a third plane, then their line of intersection perpendicular to the third plane."

Given : $(X) \cap (Y) = \overleftrightarrow{AB}$

$(X), (Y) \perp (Z)$

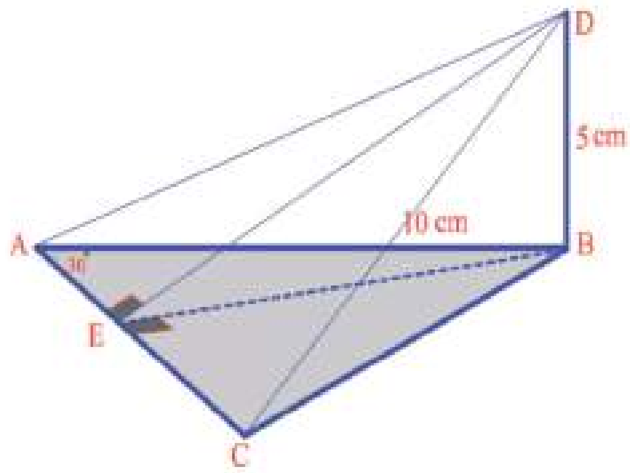
Required : $\overleftrightarrow{AB} \perp (Z)$



Proof: If $\overleftrightarrow{AB}$ is not perpendicular on Z then no more than one plane contains on (Z) (Theorem 9)

So , $\overleftrightarrow{AB} \perp (Z)$ **(Q.E.D)**

Example 1



In ΔABC

$\overline{BD} \perp (ABC)$
$m\angle A = 30^{\circ}$
$AB = 10cm,\ BD = 5cm$

Find measurement of dihedral angle $D - \overline{AC} - B$

Given :

$\overline{BD} \perp (ABC)$
$m\angle A = 30^{\circ}$
$AB = 10cm,\ BD = 5cm$

Required:

Find measurement of dihedral angle $D - \overline{AC} - B$

Proof :

In plane (ABC) ,draw $\overline{BE} \perp \overline{AC}$ at (E) point (In a plane, there is only one line perpendicular on another line at a given point)

$\because \overline{BD} \perp (ABC)$ (Given)

$\therefore \overline{DE} \perp \overline{AC}$ (Three perpendicular lines theorem)

$\angle DEB$ (Belongs to dihedral angle) $\overline{AC}$ (by definition of plane angle of dihedral angle)

$\overline{DB} \perp \overline{BE}$ (A line that is perpendicular on a plane is perpendicular on all lines contained in that plane and passing through its trace .

ΔDBE right angle at B

In ΔBEA right angle at E

$\sin 30^{\circ} = \frac{BE}{BA} \Rightarrow \frac{1}{2} = \frac{BE}{10} \Rightarrow BE = 5cm$

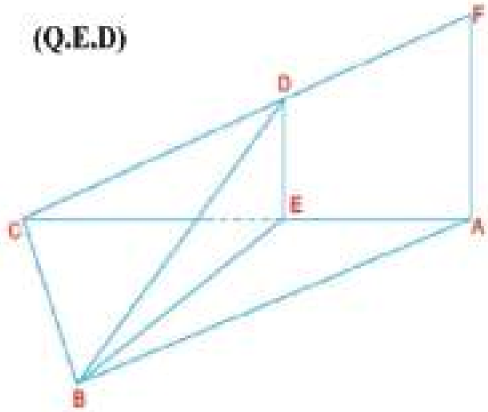
In ΔDBE which is right angle in B

$\tan(BED) = \frac{5}{5} = 1$

Measure of $m\angle BED = 45^0$

Measure of dihedral angle $D - \overline{AC} - B = 45^0$ (because measure of dihedral angle equals measure of it's plane angle and vice versa).

(Q.E.D)



Example 2

Let ABC a triangle, and let

$\overline{AF} \perp (ABC)$

$\overline{BD} \perp \overline{CF}$

$\overline{BE} \perp \overline{CA}$

Prove that:

$\overline{BE} \perp (CAF)$

$\overline{ED} \perp \overline{CF}$

Given :

$\overline{AF} \perp (ABC)$, $\overline{BE} \perp \overline{CA}$, $\overline{BD} \perp \overline{CF}$

Required:

$\overline{DE} \perp \overline{CF}$, $\overline{BE} \perp (CAF)$

Proof :

$\because$ $\overline{AF} \perp (ABC)$ **(Given)**

$\therefore$ $(CAF) \perp (ABC)$ (*Theorem 8 : two planes are orthogonal if one plane has a line perpendicular on the other)*

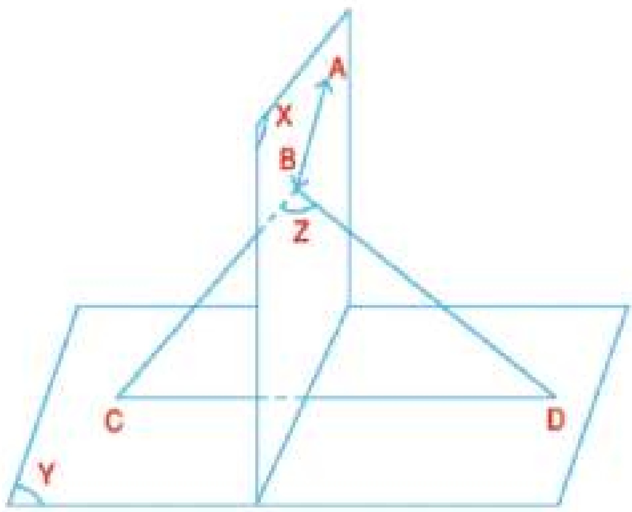
$\because$ $\overline{BE} \perp \overline{CA}$ *(Given)*

$\therefore$ $\overline{BE} \perp (CAF)$ *(Theorem 7: Let two planes are orthogonal, the line drawn on one plane and perpendicular on intercept line will be perpendicular on the other plane too).*

$\because$ $\overline{BD} \perp \overline{CF}$ *(Given)*

$\therefore$ $\overline{ED} \perp \overline{CF}$ *(Three perpendicular lines theorem)* **Q.E.D**

Example 3



(Y) , (X) two orthogonal planes

$\overline{AB} \subset (X)$

$\overleftrightarrow{CB}, \overleftrightarrow{BD}$ are perpendicular on $\overleftrightarrow{AB}$

And intersect (Y) at C,D respectively

Prove that:

$\overleftrightarrow{CD} \perp (X)$

Given:

$\overleftrightarrow{BC}, \overleftrightarrow{BD}$, $\overleftrightarrow{AB} \subset (X)$, $(X) \perp (Y)$ are perpendicular on $\overleftrightarrow{AB}$ and intersect (Y) at C,D , respectively.

Required: $\overleftrightarrow{CD} \perp (X)$

Proof: Let (Z) a plane of intersect lines $\overleftrightarrow{BC}, \overleftrightarrow{BD}$ (for each two intersect lines, there is one unique plane too contain them)

Since $\overleftrightarrow{AB} \perp \overleftrightarrow{BC}, \overleftrightarrow{BD}$ (given)

$\therefore$ $\overleftrightarrow{AB} \perp (Z)$ *(a perpendicular line on two intersect line at intersect point is perpendicular on their plane)*

$\because$ $\overleftrightarrow{AB} \subset (X)$ (Given)

$\therefore (X) \perp (Z)$ *(Two planes are perpendicular if one plane contains a line perpendicular on the other plane).*

$\because (X) \perp (Y)$ (Given)

Since $(Z) \cap (Y) = \overleftrightarrow{CD}$ *(because it is contained in both)*

$\therefore \overleftrightarrow{CD} \perp (X)$ *If each two intersect planes are perpendicular on a third plane, then, intersect line is perpendicular on the third plane.* **Q.E.D**

Exercises

1. Prove that plane of planar angle which belongs to dihedral angle is perpendicular on its edge.
2. Prove that if a line is parallel to a plane and it is perpendicular to other plane, then these two planes are perpendicular to each other.
3. Prove that if the plane is perpendicular to one of two parallel planes it is perpendicular to the other, too.
4. Four points A, B, C and D are not at the same plane where $AB = AC$, $E \in \overline{BC}$. If $\angle AED$ is plane angle of the dihedral angle $A\text{-}\overline{BC}\text{-}D$. Prove that $CD = BD$
5. If two intersecting lines are parallel to a given plane and they are each perpendicular to two intersecting planes, then the line of intersection of the two intersecting planes is perpendicular to the given plane.
6. A circle with diameter $\overline{AB}$, $\overline{AC}$ perpendicular on its plane, D is a point in the circle; prove that (CDA) is perpendicular on (CDB).

(6 - 3) The Orthogonal Projection on a Plane

1- Projection of a point on a plane: it is the trace of the column drawn at that point in the plane.

2- Projection of a set of points on a plane: Let L , a set of points in space, its projection is the set of traces of columns drawn at these points on the plane.

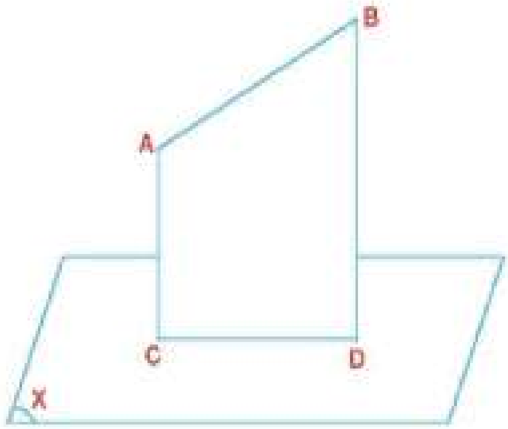
3- Projection of a non-perpendicular segment on a given plane: it is the segment defined by traces of two columns drawn at ends of the given plane.

Let $\overline{AB}$ non-perpendicular on (X) and let

$\overline{AC} \perp (X)$ projection of A on (X) is C

$\overline{BD} \perp (X)$ projection of B on (X) is D

$\therefore$ Projection of $\overline{AB}$ on (X) is $\overline{CD}$



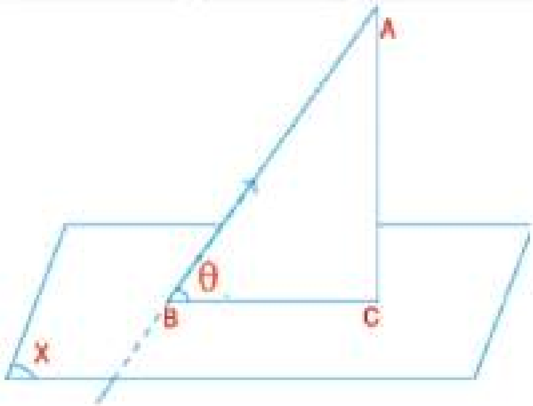
Note:

If $\overline{AB} // (X)$

Then $AB = CD$

4- Inclined Line on a plane: it is the non-perpendicular line on a plane bisects it.

5- Angle of Inclination: it is the angle bounded by the inclined and its projection on a plane.



Let $\overleftrightarrow{AB}$ inclined on (X) at B

Let $\overline{AC} \perp (X)$ in C

$\therefore$ C is A projection on (X) where $A \notin (X)$

Also, B is itself projection where $B \in (X)$

$\Rightarrow \overline{BC}$ Projection of $\overline{AB}$ on (X)

i.e. $0 < \theta < 90^0$

$\theta \in (0, 90^0)$

6- Length of Project

Length of project of a segment on a plane = length of inclined × cos of angle of inclination

When $\overline{AB}$ is inclined on (X) and angle of Inclination Θ and project $\overline{BC}$, then $BC = AB \cdot \cos\theta$

7- Project of Inclined Plane on X

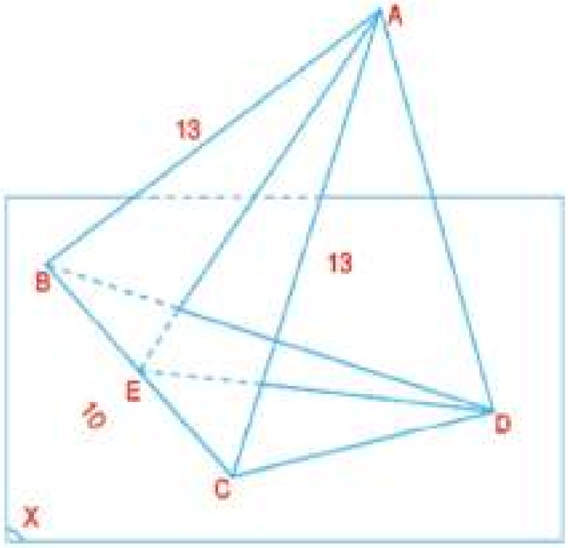
Angle of Inclination of a plane on a given plane is measurement of planar angle belonging to dihedral angle between them.

Area of project of inclined region on a given plane = area of inclined region × cos. of angle of inclination

Let A area of inclined region, A' area of project, Θ measurement of angle of an inclination $A' = A \cdot \cos\theta$

Example 4

ABC is triangle, $\overline{BC} \subset (X)$



Dihedral angle between triangle

Plane ABC and plane X,

it s measurement is 60^0 , if AB=AC=13cm, BC=10 cm

Find projection of triangle (ABC) on (X)

Then find project area ΔABC on (X)

Given:

$\Delta ABC, \overline{BC} \subset (X)$

$(ABC) - \overline{BC} - (X) = 60^0$

$AB = AC = 13, \ BC = 10$

Required:

Find project of ΔABC on (X) and find project area ΔABC on (X)

Proof:

Draw $\overline{AD} \perp (X)$ in D (a column can be drawn on a plane at a given point)

$\overline{CD}$ is project of $\overline{AC}$
$\overline{BD}$ is project of $\overline{AB}$
} (projection of a segment on a given plane is the segment bounded by traces of columns drawn on plane at ends of segment)

$\overline{BC}$ is project of itself on (X)

$\therefore$ ΔBCD is project of ΔABC on (X)

In (ABC) we draw $\overline{BC} \perp \overline{AE}$ in E (In one plane, a perpendicular line can be drawn on another line from a given point)

Since AC=AB (given)

$\therefore$ EC=BE=5cm (The perpendicular column from vertex of isosceles at the base, divides the base into half).

$\therefore \overline{ED} \perp \overline{BC}$ (Postulate of three columns theorem)

$\therefore$ $\angle DEA$ belongs to dihedral $\overline{BC}$ (by definition of belonging angle).

However, measurement of dihedral angle is $BC = 60^0$ (given)

In ΔAEB right in E

$$AE = \sqrt{169 - 25} = \sqrt{144} = 12cm$$

In ΔAED right in D

$$\cos 60^0 = \frac{ED}{AE} \Rightarrow \frac{1}{2} = \frac{ED}{12} \Rightarrow ED = 6cm$$

Area of triangle BCD $= \frac{1}{2} \cdot (10) \quad (6) = 30cm^2$ Q.E.D

Note

If area of project is required, it can be found as follows

Area of BCD = area of ABC $\times \cos 60^0$

$$\frac{1}{2} \cdot \left((12)(10)\left(\frac{1}{2}\right) \right) = 30cm^2$$

Exercises

1. Prove that length of segment parallel to known plane equals length of its projection on known plane and parallel to it
2. Prove that If two parallel planes cut by a line, then its slope on one of them equals to its slope on the other.
3. If parallel lines intersect a plane they are equally inclined the plane. Prove that.
4. If two different inclined lines are drawn from a point, which does not belong to a given plane , then taller one has a slope angle on the plane smaller than the other angle on it.
5. If two inclined lines are drawn from any point to a plane, then one with smaller slope has longer length.
6. Prove that angle of inclination between line and its projection on a plane is smaller than the angle bounded by line itself and any other line drawn from its position within that plane.

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