Ripublic Of Iraq
Ministry Of Education
General Directorato Of Curricula

MATHEMATICS

SCIENTIFIC SECONDARY

5^{TH}

2023 **Eleventh Eddition**

Scintific Supervisor

Hussein Sadik AL-Allak

Art Supervisor

Bashar Hamed AL- wan

الموقع والصفحة الرسمية للمديرية العامة للمناهج

www.manahj.edu.iq

manahjb@yahoo.com

Info@manahj.edu.iq

manahjb

manahj

المديرية العامة للمناهج

قسم التحضير الطباعي

CONTENTS

Chapter – 5 LIMIT

A – Limit of a Polynomial Function
B – One Sided Limits
C – Limits of Spherical Functions
D – Continuity

Chapter - 6 DERIVATIONS

A – Tangents
B – Derivative of a Function
C – Left Hand and Right Hand Derivations
D – Differentiability and continuity

TECHNIQUES of DIFFERENTIATION

A – Basic Differentiation Rules
B – The Product and the Quotient Rules
C – The Chain Rule
D – Derivatives of Trigonometric Function
E – Implicit Differentiation

Chapter - 7 SPACE GEOMETRY

Axioms of space gometry

Theorems

Chapter - 8 COMBINATION AND PERMUTATION and probaility

PERMUTATIONS

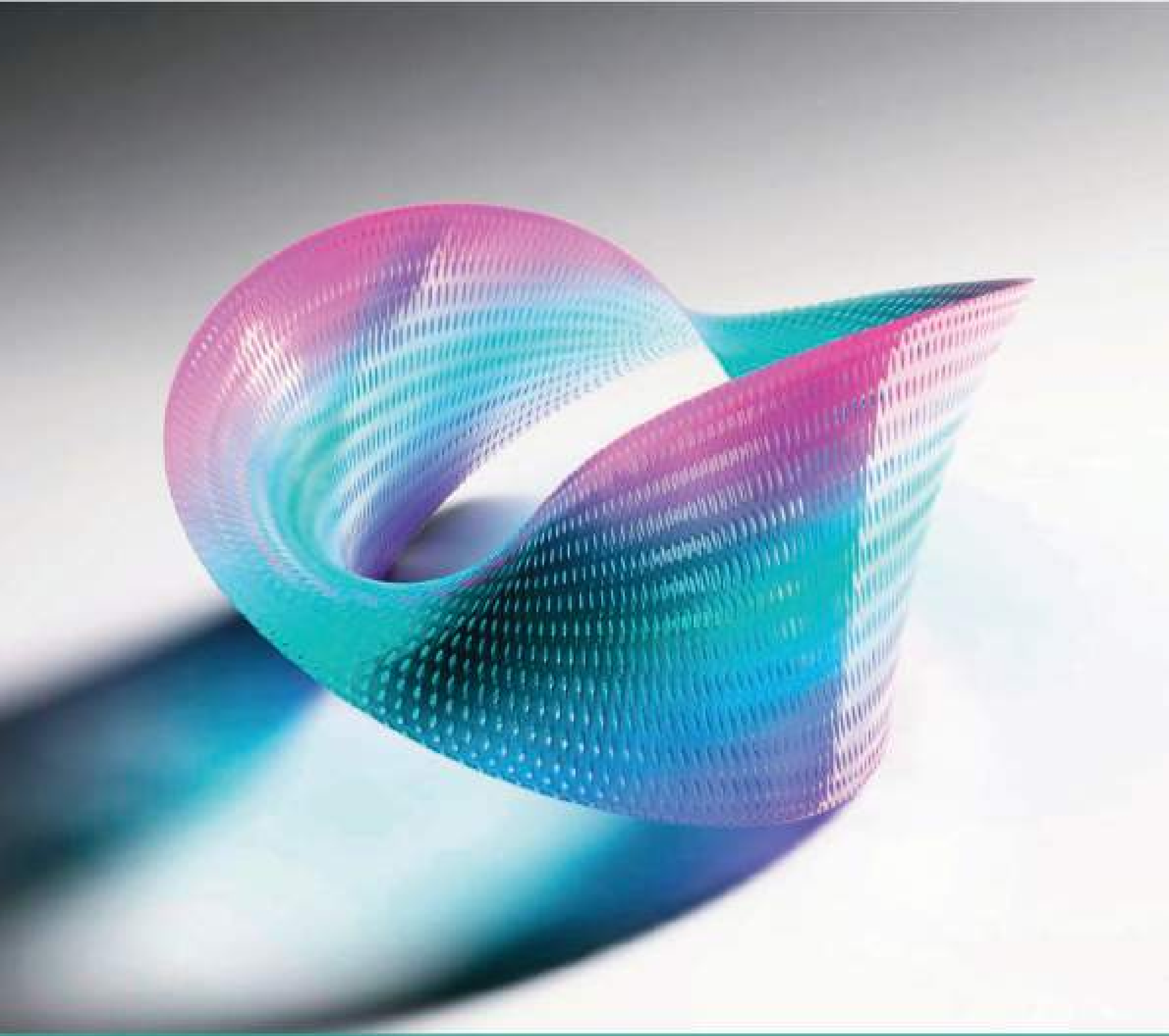
A – Factorial Notation
B – Permutations of "r" Elements Selected from "n"

COMBINATIONS

A – Combinations of "r" Elements Selected from "n" Ele

Chapter – 9 MATRICES

A- Basic Concept
B – Types of Matrix
C – Operations on Matrices
D – Determnal
E- Gramer's Rule



Chapter 1

LOGARITHM

LOGARITHM

Consider the relation $a^x = N$. Imagine that we are asked to find one of the three numbers a, x or N given the other two numbers. Three examples of this are shown below.

Case	Solution	Method
$2^5 = p$	32	raise 2 to the 5th power
$p^3 = 27$	3	take the 3rd root of 27
$3^p = 5$	?	?

We can see that we cannot solve the last example with the algebra we have studied so far. We need to introduce a new concept: a **logarithm**.

A. BASIC CONCEPT

The **logarithm** of a number N to a base a is the power to which a must be raised in order to obtain N. We write this as $\log_a N$. In other words,

$$a^{\log_a N} = N \text{ where } a^x = N \text{ and } x = \log_a N.$$

This equation is called the **fundamental identity of logarithms**. In this equation, the base of the logarithm (a) is always positive and different from 1, and the number whose logarithm is taken (N) is positive. In other words, **negative numbers and zero do not have logarithms**.

Definition

logarithm, argument, base, exponential form, logarithmic form

For $a > 0$, $a \neq 1$ and $x > 0$, the real number y which is defined by

$$y = \log_a x \Leftrightarrow a^y = x$$

is called the **logarithm of x to the base a**. In this notation, x is called the **argument** of the logarithm.

We say that the equation $a^y = x$ is in **exponential form** and $\log_a x = y$ is the same equation in **logarithmic form**.

EXAMPLE 1 Write the equalities in logarithmic form.

a. $2^3 = 8$ b. $5^0 = 1$ c. $3^{-2} = \frac{1}{9}$

Solution By the definition of a logarithm, $a^y = x \Leftrightarrow y = \log_a x$. Therefore,

a. $2^3 = 8 \Leftrightarrow 3 = \log_2 8$.   b. $5^0 = 1 \Leftrightarrow 0 = \log_5 1$.

c. $3^{-2} = \frac{1}{9} \Leftrightarrow -2 = \log_3 \frac{1}{9}$.

EXAMPLE 2 Write the equalities in exponential form.

a. $\log_{10} 100 = 2$ b. $\log_3 \frac{1}{27} = -3$ c. $\log_2 1 = 0$

Solution Again we use the definition $\log_a x = y \Leftrightarrow x = a^y$.

a. $\log_{10} 100 = 2 \Leftrightarrow 100 = 10^2$ b. $\log_3 \frac{1}{27} = -3 \Leftrightarrow \frac{1}{27} = 3^{-3}$

c. $\log_2 1 = 0 \Leftrightarrow 1 = 2^0$

EXAMPLE 3 Solve each equation for x.

a. $\log_x 27 = 3$ b. $\log_4 x = \frac{1}{2}$ c. $\log_4 16 = x$

Solution

a. $\log_x 27 = 3 \Leftrightarrow x^3 = 27 \Leftrightarrow x = 3$ b. $\log_4 x = \frac{1}{2} \Leftrightarrow 4^{\frac{1}{2}} = x \Leftrightarrow x = 2$

c. $\log_4 16 = x \Leftrightarrow 4^x = 16 \Leftrightarrow 4^x = 4^2 \Leftrightarrow x = 2$

EXAMPLE 4 Calculate the logarithms.

a. $\log_2 4$ b. $\log_3 \frac{1}{9}$ c. $\log_2(\log_3 9)$

Solution a. Let $\log_2 4 = y$.

Then $\log_2 4 = y \Leftrightarrow 2^y = 4 \Leftrightarrow 2^y = 2^2 \Leftrightarrow y = 2$, so $\log_2 4 = 2$.

b. Similarly, $\log_3 \frac{1}{9} = y \Leftrightarrow 3^y = \frac{1}{9} \Leftrightarrow 3^y = 3^{-2} \Leftrightarrow y = -2$.

c. Let $\log_3 9 = m$. Then $3^m = 9 \Leftrightarrow 3^m = 3^2 \Leftrightarrow m = 2$. So we need to calculate $\log_2 2$. Starting with $\log_2 2 = n$, we get $2^n = 2$ which gives us $n = 1$. Thus, $\log_2(\log_3 9) = 1$.

> **Remember:**
> $a^x = a^y \Leftrightarrow x = y$
> by the bijective property of exponential functions.

EXAMPLE 5 Calculate the logarithms.

a. $\log_3 \frac{1}{3}$ b. $\log_{\frac{1}{3}} \sqrt[3]{81}$ c. $\log_a \sqrt[3]{a\sqrt{a}}$ d. $\log_3(\log_2(\log_9 81))$

Solution

a. By the definition of a logarithm, we can write $\log_3 \frac{1}{3} = y \Leftrightarrow 3^y = \frac{1}{3} \Leftrightarrow 3^y = 3^{-1} \Leftrightarrow y = -1$.

So $\log_3 \frac{1}{3} = -1$.

b. $\log_{\frac{1}{3}} \sqrt[3]{81} = y \Leftrightarrow (\frac{1}{3})^y = (81)^{\frac{1}{3}} \Leftrightarrow 3^{-y} = (3^4)^{\frac{1}{3}} \Leftrightarrow 3^{-y} = 3^{\frac{4}{3}} \Leftrightarrow y = -\frac{4}{3}$

c. $\log_a \sqrt[3]{a\sqrt{a}} = y \Leftrightarrow a^y = (a\sqrt{a})^{\frac{1}{3}} \Leftrightarrow a^y = (a \cdot a^{\frac{1}{2}})^{\frac{1}{3}} \Leftrightarrow a^y = (a^{\frac{3}{2}})^{\frac{1}{3}} \Leftrightarrow a^y = a^{\frac{1}{2}} \Leftrightarrow y = \frac{1}{2}.$

So $\log_a \sqrt[3]{a\sqrt{a}} = \frac{1}{2}.$

d. Starting from the innermost logarithm, we have

$\log_9 81 = x \Leftrightarrow 9^x = 81 \Leftrightarrow 9^x = 9^2 \Leftrightarrow x = 2.$

So we have to calculate $\log_3 (\log_2 2)$, and $\log_2 2 = y \Leftrightarrow 2^y = 2 \Leftrightarrow y = 1.$

So the given expression becomes $\log_3 1$, which is equal to zero:

$\log_3 1 = z \Leftrightarrow 3^z = 1 \Leftrightarrow z = 0.$ In conclusion, $\log_3(\log_2(\log_9 81)) = 0.$

Notice that in these examples we were able to find the desired logarithm by writing the argument as a rational power of the base. This is not always possible, however: many logarithms (for example: $\log_2 3$ and $\log_3 5$) are irrational, and cannot be calculated in this way.

EXAMPLE 6 Evaluate the expressions.

a. $2^{\log_2 8}$ b. $25^{\log_5 3}$ c. $3^{3\log_3 2}$

Solution a. By the fundamental identity of logarithms, $a^{\log_a N} = N$ and so $2^{\log_2 8}$ will be equal to 8. However, let us try to evaluate the expression in a different way:

Let $\log_2 8 = t$. Then we have to calculate 2^t.

By definition we have $\log_2 8 = t \Leftrightarrow 2^t = 8$. So $2^{\log_2 8} = 8.$

b. Let $\log_5 3 = t$. Then we have to calculate 25^t. By definition, $\log_5 3 = t \Leftrightarrow 5^t = 3$. So $25^t = (5^2)^t = (5^t)^2 = (3)^2 = 9$, i.e. $25^{\log_5 3} = 9.$

$(a^m)^n = (a^n)^m = a^{m \cdot n}$

c. In a similar way, let $\log_3 2 = t$ and let us calculate 3^{3t}. By definition, $\log_3 2 = t \Leftrightarrow 3^t = 2$, i.e. $3^{3t} = (3^t)^3 = 2^3 = 8$. So $2^{3\cdot\log_3 2} = 8.$

Check Yourself

1. Write the equalities in logarithmic form.

a. $2^4 = 16$ b. $10^3 = 1000$ c. $3^0 = 1$ d. $125^{\frac{1}{3}} = 5$ e. $3^{-3} = \frac{1}{27}$ f. $(2\sqrt{2})^{-\frac{2}{3}} = \frac{1}{2}$

2. Write the equalities in exponential form.

a. $\log_{10} 0.01 = -2$ b. $\log_{\frac{1}{2}} \frac{1}{16} = 4$ c. $\log_{10} 10000 = 4$

d. $\log_3 \frac{1}{81} = -4$ e. $\log_2 32 = 5$ f. $\log_{\frac{1}{5}} 125 = -3$

3. State whether each statement is true or false.

a. $\log_3 729 = 6$ b. $\log_{\frac{1}{2}} \sqrt[3]{4} = -\frac{2}{3}$ c. $\log_{10} \frac{1}{10\sqrt{10}} = -\frac{3}{2}$

d. $\log_{3\sqrt{3}} \frac{1}{3} = -\frac{3}{2}$ e. $\log_a \sqrt{a\sqrt{a\sqrt{a}}} = \frac{7}{8}$ $(a > 0,\ a \neq 1)$

4. Determine the logarithms of each set of numbers to the given base.

a. $27,\ 1,\ \frac{1}{9},\ \frac{1}{\sqrt[3]{3}},\ \sqrt[3]{9},\ \frac{9}{\sqrt[4]{3}}$ to base 3 b. $2,\ 4,\ 1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{32},\ 32,\ -64$ to base $\frac{1}{2}$

5. Solve for x.

a. $\log_x 4 = 2$ b. $\log_4 x = -\frac{1}{2}$ c. $\log_{25} 125 = x$

6. Calculate the logarithms.

a. $\log_5 25$ b. $\log_6 1296$ c. $\log_{49} \frac{1}{7}$ d. $\log_3(\log_2(\log_2 256))$

7. Evaluate the expressions.

a. $3^{-\log_3 4}$ b. $(2^{\log_2 5})^2$ c. $25^{-\log_5 10}$ d. $49^{\frac{1}{2}\log_7 \frac{1}{4}}$ e. $2^{2\log_2 5+\log_2 3}$

Answers

1. a. $\log_2 16 = 4$ b. $\log_{10} 1000 = 3$ c. $\log_3 1 = 0$

d. $\log_{125} 5 = \frac{1}{3}$ e. $\log_3 \frac{1}{27} = -3$ f. $\log_{2\sqrt{2}} \frac{1}{2} = -\frac{2}{3}$

2. a. $10^{-2} = 0.01$ b. $(\frac{1}{2})^4 = \frac{1}{16}$ c. $10^4 = 10000$

d. $3^{-4} = \frac{1}{81}$ e. $2^5 = 32$ f. $(\frac{1}{5})^{-3} = 125$

3. a. true b. true c. true d. false e. true

4. a. $3, 0, -2, -\frac{1}{3}, \frac{2}{3}, \frac{7}{4}$ b. $-1, -2, 0, 1, 2, 5, -5$, undefined

5. a. 2 b. $\frac{1}{2}$ c. $\frac{3}{2}$

6. a. 2 b. 4 c. $-\frac{1}{2}$ d. 1

7. a. $\frac{1}{4}$ b. 25 c. $\frac{1}{100}$ d. $\frac{1}{4}$ e. 75

B. TYPES OF LOGARITHM

1. Common Logarithms

Our counting system is based on the number 10. For this reason, a lot of logarithmic work uses the base 10. Logarithms to the base 10 are called **common logarithms**. We often write **log** x or to mean $\log_{10} x$. In this module, we will use $\log x$ to mean $\log_{10} x$.

Common logarithms are widely used in computation. Mathematicians have compiled extensive and highly accurate tables of common logarithms for use in these calculations. These tables and their use will be discussed later in this module.

2. Natural Logarithms

$e = 2.71828...$

Logarithms to the base e are called **natural logarithms** or **Euler logarithms**. We often write $\ln x$ to mean the natural logarithm $\log_e x$.

Natural logarithms are widely used in mathematical analysis in the study of limits, derivatives and integrals.

C. PROPERTIES OF LOGARITHMS

Property 1

If the argument and the base of a logarithm are equal, the logarithm is equal to 1. Conversely, if the logarithm is 1 then the argument and the base are equal:

$$a = b \Leftrightarrow \log_a b = 1 \quad (a > 0,\ a \neq 1),\ b > 0$$

Proof By the fundamental identity of logarithms we have $a^{\log_a N} = N$. Setting $N = a$ gives us $a^{\log_a a} = a = a^1$, which gives us $\log_a a = 1$.

For example, $\log_3 3 = 1$, $\log 10 = 1$, $\ln e = 1$ and $\log_{\frac{1}{2}} \frac{1}{2} = 1$.

Property 2

The logarithm of 1 to any base is zero:

$$\log_a 1 = 0.$$

Proof $a^{\log_a 1} = 1 = a^0$. So $a^{\log_a 1} = a^0$, which gives us $\log_a 1 = 0$.

For example, $\log_3 1 = 0$, $\log_{\frac{1}{2}} 1 = 0$ and $\log_\pi 1 = 0$.

Property 3

The logarithm of the product of two or more positive numbers to a given base is equal to the sum of the logarithms of the numbers to that base:

$$\log_a(x \cdot y) = \log_a x + \log_a y \qquad (x, y > 0).$$

Proof $a^{\log_a(x \cdot y)} = x \cdot y$. Substituting $x = a^{\log_a x}$ and $y = a^{\log_a y}$ gives us

$a^{\log_a(x \cdot y)} = a^{\log_a x} \cdot a^{\log_a y} = a^{\log_a x + \log_a y}$.

Comparing the exponents of the expressions on both sides gives us the required equation:

$$\log_a(x \cdot y) = \log_a x + \log_a y.$$

Be careful!
$\log_a(x + y) \neq \log_a x + \log_a y$

For example,

$\log_2 6 = \log_2(2 \cdot 3) = \log_2 2 + \log_2 3 = 1 + \log_2 3$

$\log_3 30 = \log_3(3 \cdot 10) = \log_3 3 + \log_3 10 = 1 + \log_3 10$

$\log_3 30 = \log_3(6 \cdot 5) = \log_3 6 + \log_3 5$

$\log_2 5 + \log_2 3 = \log_2(5 \cdot 3) = \log_2 15.$

Notice that we can generalize this property as follows:

$$\log_a(x_1 \cdot x_2 \cdot x_3 \cdot \ldots \cdot x_k) = \log_a x_1 + \log_a x_2 + \ldots + \log_a x_k \quad (x_1, x_2, x_3, \ldots, x_k > 0).$$

For example, we can write

$\log_2 30 = \log_2(2 \cdot 3 \cdot 5) = \log_2 2 + \log_2 3 + \log_2 5 = 1 + \log_2 3 + \log_2 5.$

EXAMPLE 7 Calculate $\log_4 2 + \log_4 8$.

Solution $\log_4 2 + \log_4 8 = \log_4(2 \cdot 8) = \log_4(4 \cdot 4) = \log_4 4 + \log_4 4 = 1 + 1 = 2$

EXAMPLE 8 Calculate $\log_2 3 + \log_2 5 + \log_2 \frac{1}{15}$.

Solution $\log_2 3 + \log_2 5 + \log_2 \frac{1}{15} = \log_2(3 \cdot 5 \cdot \frac{1}{15}) = \log_2 1 = 0$

Property 4 The logarithm of the power of a positive number is equal to the product of the power and the logarithm of the number.

$$\log_a(x^m) = m \cdot \log_a x \quad (m \in R,\ x > 0).$$

Be careful!
$(\log_a x)^m \neq m \cdot \log_a x$

Proof $x^m = a^{\log_a(x^m)}$. After substituting $x = a^{\log_a x}$ on the left side, we get $(a^{\log_a x})^m = a^{\log_a(x^m)}$, which gives us $a^{m \cdot \log_a x} = a^{\log_a(x^m)}$. Since the bases are the same on both sides, we can conclude $m \cdot \log_a x = \log_a(x^m)$.

For example,

$(a^m)^n = a^{m \cdot n}$

$$\log_2 8 = \log_2(2^3) = 3 \cdot \log_2 2 = 3 \cdot 1 = 3$$
$$\log_3 \frac{1}{243} = \log_3 \frac{1}{3^5} = \log_3(3^{-5}) = -5 \cdot \log_3 3 = -5 \cdot 1 = -5$$
$$\log_2 \sqrt{125} = \log_2 \sqrt{5^3} = \log_2 (5^{\frac{3}{2}}) = \frac{3}{2} \cdot \log_2 5.$$

$\sqrt[m]{x^n} = x^{\frac{n}{m}}$

$\frac{1}{x^n} = x^{-n}$

Note

This property gives us the following special cases:

4a. $\log_a \frac{1}{x^n} = -n \cdot \log_a x$ 4b. $\log_a \sqrt[m]{x^n} = \frac{n}{m} \cdot \log_a x.$

EXAMPLE 9 Write each sum as a single logarithm.

a. $(2 \cdot \log_3 a) + (3 \cdot \log_3 b) - \log_3 c$ b. $(\frac{1}{2} \cdot \log_2 a) + (3 \cdot \log_2 b) - (\frac{3}{2} \cdot \log_2 c)$

Solution We apply the property $\log_a(x^m) = m \cdot \log_a x$.

a. $(2 \cdot \log_3 a) + (3 \cdot \log_3 b) - \log_3 c = \log_3(a^2) + \log_3(b^3) + (-1) \cdot \log_3 c \Leftrightarrow$

$\log_3(a^2) + \log_3(b^3) + \log_3(c^{-1}) = \log_3(a^2 \cdot b^3 \cdot c^{-1}) = \log_3(\frac{a^2 \cdot b^3}{c})$

b. $(\frac{1}{2} \cdot \log_2 a) + (3 \cdot \log_2 b) - (\frac{3}{2} \cdot \log_2 c) = \log_2 a^{\frac{1}{2}} + \log_2 b^3 + (-\frac{3}{2}) \cdot \log_2 c \Leftrightarrow$

$\log_2 \sqrt{a} + \log_2 b^3 + \log_2 c^{(-\frac{3}{2})} = \log_2 \sqrt{a} + \log_2 b^3 + \log_2(\frac{1}{\sqrt{c^3}}) \Leftrightarrow$

$\log_2(\sqrt{a} \cdot b^3 \cdot \frac{1}{\sqrt{c^3}}) = \log_2(\frac{\sqrt{a} \cdot b^3}{\sqrt{c^3}})$

EXAMPLE 10 Calculate $\log_2 \sqrt[4]{2 \cdot \sqrt{8 \cdot \sqrt[3]{16}}}$.

Solution

$$\log_2 \sqrt[4]{2 \cdot \sqrt{8 \cdot \sqrt[3]{16}}} = \log_2 \sqrt[4]{2 \cdot \sqrt{8 \cdot 16^{\frac{1}{3}}}} = \log_2 \sqrt[4]{2 \cdot \sqrt{2^3 \cdot (2^4)^{\frac{1}{3}}}}$$

$$= \log_2 \sqrt[4]{2 \cdot \sqrt{2^{3+\frac{4}{3}}}} = \log_2 \sqrt[4]{2 \cdot \sqrt{2^{\frac{13}{3}}}} = \log_2 \sqrt[4]{2 \cdot (2^{\frac{13}{3}})^{\frac{1}{2}}} = \log_2 \sqrt[4]{2^{1+\frac{13}{6}}}$$

$$= \log_2 \sqrt[4]{2^{\frac{19}{6}}} = \log_2(2^{\frac{19}{24}}) = \frac{19}{24} \cdot \underbrace{\log_2 2}_{1} = \frac{19}{24}$$

Property 5 The logarithm of the quotient of two positive numbers is equal to the difference between the logarithms of the dividend and the divisor to the same base:

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y.$$

> $\dfrac{\log_a x}{\log_a y} \neq \log_a x - \log_a y$

Proof $\dfrac{x}{y} = a^{\log_a\left(\frac{x}{y}\right)}$. If we substitute $x = a^{\log_a x}$ and $y = a^{\log_a y}$ on the left side, we obtain

$$\frac{a^{\log_a x}}{a^{\log_a y}} = a^{\log_a\left(\frac{x}{y}\right)} \Leftrightarrow a^{\log_a x - \log_a y} = a^{\log_a\left(\frac{x}{y}\right)} \Leftrightarrow \log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y.$$

> $\dfrac{a^m}{a^n} = a^{m-n}$

For example,

$$\log_2 \frac{5}{3} = \log_2 5 - \log_2 3$$

$$\log_5(0.12) = \log_5\left(\frac{12}{100}\right) = \log_5\left(\frac{3}{25}\right) = \log_5 3 - \log_5(5^2) = \log_5 3 - 2$$

$$\log_2 10 + \log_2 4 - \log_2 5 = \log_2(10 \cdot 4) - \log_2 5 = \log_2 40 - \log_2 5 = \log_2 \frac{40}{5} = \log_2 8 = 3.$$

Notice that we can combine properties 4 and 5 to write expressions with addition and subtraction of logarithms as the logarithm of a single fraction. The addends form the numerator of the fraction and the subtrahends form the denominator, for example:

$$\log_a b + \log_a c - \log_a d + \log_a e - \log_a f = \log_a\left(\frac{b \cdot c \cdot e}{d \cdot f}\right).$$

As a numerical example, consider

$$\log_3 15 - \log_3 5 + \log_3 6 - \log_3 2 = \log_3\left(\frac{15 \cdot 6}{5 \cdot 2}\right) = \log_3 9 = \log_3(3^2) = 2.$$

Remember that this property only applies to logarithms with a common base.

EXAMPLE 11 Express log 30 and log $3.\bar{3}$ in terms of p given $\log 3 = p$.

Solution Since $30 = 3 \cdot 10$, we get $\log 30 = \log(3 \cdot 10) = \underbrace{\log 3}_{p} + \underbrace{\log 10}_{1} = p + 1$.

Since $3.\bar{3} = \frac{10}{3}$, we have $\log 3.\bar{3} = \log\frac{10}{3} = \log 10 - \log 3 = 1 - p$.

EXAMPLE 12 Given $\log 300 = 2.47712$, calculate $\log(0.0027)$.

Solution

$\log(0.0027) = \log(\frac{27}{10^4}) = \log 27 - \log 10^4 = \log 3^3 - 4 \cdot \log 10$

$= (3 \cdot \log 3) - 4$ (1)

$\log 300 = \log(3 \cdot 100) = \log 3 + \log 10^2 = \log 3 + 2 \cdot \log 10 = 2 + \log 3$. So $\log 3 = \log 300 - 2$.

Using $\log 300 = 2.47712$, we get $\log 3 = 2.47712 - 2 = 0.47712$. (2)

Combining (1) and (2) gives us $\log(0.0027) = (3 \cdot 0.47712) - 4 = -2.56864$.

$\log 10 = \log_{10} 10 = 1$

EXAMPLE 13 Write each logarithm as a sum or difference of logarithms to base a.

a. $\log_a \frac{b^3 c^2}{d^4 e^5}$ b. $\log_a \frac{\sqrt[5]{(b+c)^2}}{(d-e)^3}$

Solution

a. $\log_a(\frac{b^3 \cdot c^2}{d^4 \cdot e^5}) = \log_a b^3 + \log_a c^2 - \log_a d^4 - \log_a e^5 = 3\log_a b + 2\log_a c - 4\log_a d - 5\log_a e$

b. We have $\log_a \frac{\sqrt[5]{(b+c)^2}}{(d-e)^3} = \log_a \sqrt[5]{(b+c)^2} - \log_a (d-e)^3 = \log_a (b+c)^{\frac{2}{5}} - 3\log_a(d-e)$

$= \frac{2}{5}\log_a(b+c) - 3\log_a(d-e)$.

Notice that logarithms cannot be distributed over addition or subtraction, and also that logarithms enable us to perform simpler operations (addition and subtraction) instead of multiplication and division. This is why logarithms are so useful in computation.

EXERCISES

A. Basic Concept

1. Calculate the logarithms.

   a. $\log_{4/3}\frac{1}{3}$ b. $\log_2\frac{1}{8}$ c. $\ln e$

   d. $\log_{1/8} 1$ e. $\log_{\sqrt{2}} 2$ f. $\log\sqrt[3]{1000}$

   g. $\log 1$ h. $\log(\ln e)$ i. $\ln\sqrt[3]{e}$

   j. $\log_3(9\ln e^3)$ k. $\ln(\log 10^e)$ l. $\ln(\log 10)$

2. Solve each equation for x.

   a. $3^x = 4$ b. $2^{x+1} = 3$ c. $3^{1-\frac{x}{2}} = 2$

   d. $\sqrt{e^x} = 4$ e. $10^x = 5$ f. $10^{x-1} = 2$

3. Simplify the expressions.

   a. $e^{\ln x}$ b. $10^{\log 3}$ c. $4^{\log_2 7}$

   d. $5^{-\log_{25} 2}$ e. $27^{\log_{1/3} 4}$ f. $(x^{\log_x 5})^{\log_5 3}$

   g. $x^{\log_x 3} + y^{\log_{1/y} 1/3} + z^{\log_z 1/3} - t^{\log_{1/t} 1/3}$

   h. $-\log_2(\log_3\sqrt{\sqrt[4]{3}})$ i. $(\frac{16}{25})^{\log_{\frac{125}{64}} 3}$

B. Types of Logarithm

4. Calculate the logarithms, using $\log 2 = 0.30103$ and $\log 3 = 0.4771$.

   a. $\log 18$ b. $\log 30$ c. $\log\frac{1}{5}$

5. Find the number of digits in each number if $\log 2 = 0.30103$ and $\log 3 = 0.4771$.

   a. 2^{50} b. 9^{10} c. 27^{9} d. 18^{20}

C. Properties of Logarithms

6. Write each expression as a single logarithm.

   a. $\frac{1}{3}\log x - \log y + \log z^2$

   b. $-\frac{1}{2}\log x + \frac{1}{2}\log y + \frac{1}{2}\log z$

7. Write each expression as the sum or difference of the logarithms of a, b and c.

a. $\log(a^3b^2c)$ b. $\log(\sqrt[3]{a}\sqrt{bc})$

8. Evaluate the expressions.

a. $\log_{24} 4 + \log_{24} 6$

b. $\log 8 + \log 25 + \log 5$

c. $\log_{1/2}\frac{1}{4} + \log_5 625 + \log_{1/3} 81 + \log_{49}\frac{1}{7}$

d. $\log_2 1000 - \log_2 125$

9. Calculate each logarithm in terms of the variable(s) provided, using the given relation(s).

a. $\log_2 3$; $\log_3 2 = a$ b. $\log 25$; $\log 2 = a$

c. $\log_7 21$; $\log_3 7 = p$ d. $\log_3 18$; $\log_3 12 = a$

e. $\log_{12} 60$; $\log_6 30 = a$ and $\log_{15} 24 = b$

10. $\log_x y = a$ is given. Express each logarithm in terms of a.

a. $\log_{x^2y^2} x^2y^3$ b. $\log_{x/y^2} x^3y^4$

11. Simplify the expressions.

a. $(\log_3 625 \cdot \log_{1/5} 9) + (\log_4 \frac{1}{125} \cdot \log_{1/25} 1024)$

b. $\log_{a^2} b^3 \cdot \log_{b^3} c^4 \cdot \log_{c^4} d^5 \cdot \log_{d^5} a$

12. Find x in each case.

a. $\log_2 x = 3 - (2 \cdot \log_2 3) + (3 \cdot \log_2 5)$

b. $\log_3 x = 2 + (3 \cdot \log_3 5) - (2 \cdot \log_3 4)$

13. Prove each equality.

a. $\log_{x_1} x_2 \cdot \log_{x_2} x_3 \cdot \ldots \cdot \log_{x_n} x_1 = 1$

b. $x^{\log\frac{y}{z}} \cdot y^{\log\frac{z}{x}} \cdot z^{\log\frac{x}{y}} = 1$

14. Show that if $a^2 + b^2 = 7ab$ $(a, b > 0)$ then

$$\log\frac{a+b}{3} = \frac{1}{2}(\log a + \log b).$$

Chapter 2

SEQUENCE

INTRODUCTION . . .

An interesting unsolved problem in mathematics concerns the 'hailstone sequence', which is defined as follows: Start with any positive integer. If that number is odd, then multiply it by three and add one. If it is even, divide it by two. Then repeat. For example, starting with the number 10 we get the hailstone sequence 10, 5, 16, 8, 4, 2, 1, 4, 2, 1... . Some mathematicians have *conjectured* (guessed) that no matter what number you start with, you will always reach 1. This conjecture has been found true for all starting values up to 1,200,000,000,000. However, the conjecture, which is known as the 'Collatz Problem', '3n+1 Problem', or 'Syracuse Algorithm', still has not been proved true for all numbers.

Number sequences have been an interesting area for all mathematicians throughout history. Geometric sequences appear on Babylonian tablets dating back to 2100 BC. Arithmetic sequences were first found in the Ahmes Papyrus which is dated at 1550 BC. The reason behind the names 'arithmetic' and 'geometric' is that each term in a geometric (or arithmetic) sequence is the geometric (or arithmetic) mean of its successor and predecessor. If we think of a rectangle with side lengths x and y, then the geometric mean $\sqrt{xy}$ is the side length of a square that has the same area as this rectangle. Finding the dimensions of a square with the same area as a given rectangle was considered in those days as a very geometric problem. Although the arithmetic mean $(x + y)/2$ can also be interpreted geometrically (it is the length of the sides of a square having the same perimeter as the rectangle), lengths were viewed more as arithmetic, because it is easier to handle lengths by addition and subtraction, without having to think about two-dimensional concepts such as area. Although both problems involve arithmetic and can be interpreted geometrically, in ancient times one was viewed as much more geometric than the other, therefore the names.

Zeno (490-425 B.C.) was a mathematician whose paradoxes about motion puzzled mathematicians for centuries. They involved the sum of an infinite number of positive terms to a finite number. Zeno wasn't the only ancient mathematician to work on sequences. Several of the ancient Greek mathematicians used sequences to measure areas and volumes of shapes and regions. By using his reasoning technique called the 'method', Archimedes (287-212 B.C.) constructed several examples and tried to explain how infinite sums could have finite results. Among his many results was that the area under a parabolic arc is always two-thirds the base times the height.

The next major contributor to this area of mathematics was Fibonacci (1170-1240). He discovered a sequence of integers in which each number is equal to the sum of the preceding two numbers (1, 1, 2, 3, 5, 8, ...), and introduced it as a model of the breeding population of rabbits. This sequence has many remarkable properties and continues to find applications in many areas of modern mathematics and science. During this same period, Chinese astronomers developed numerical techniques to analyze their observation data and used the idea of finite differences to help analyze trends in their data.

Oresme (1325-1382) studied rates of change, such as velocity and acceleration, using sequences. Two hundred years later, Stevin (1548-1620) understood the physical and mathematical conceptions of acceleration due to gravity using series and sequences. During that time Galileo (1564-1642) applied mathematics to the sciences, especially astronomy. Based on his study of Archimedes, Galileo improved our understanding of hydrostatics. He developed equations for free-fall motion under gravity and the motion of the planets. Up until the middle of the 17th century, mathematicians developed and analyzed series of numbers.

Newton (1642-1727) and Leibnitz (1646-1716) developed several series representations for functions. Maclaurin (1698-1746), Euler (1707-1783), and Fourier (1768-1830) often used infinite series to develop new methods in mathematics. Sequences and series have become standard tools for approximating functions and calculating results in numerical computing.

The self-educated Indian mathematician Srinivasa Ramanujan (1887-1920) used sequences and power series to develop results in number theory. Ramanujan's work was theoretical and produced many important results used by mathematicians in the 20th century.

1 REAL NUMBER SEQUENCES

Real number sequences are strings of numbers. They play an important role in our everyday lives. For example, the following sequence:

$$20, 20.5, 21, 22, 23.4, 23.6, \ldots$$

gives the temperature measured in a city at midday for five consecutive days. It looks like the temperature is rising, but it is not possible to exactly predict the future temperature.

The sequence:

$$64, 32, 16, 8, \ldots$$

is the number of teams which play in each round of a tournament so that at the end of each game one team is eliminated and the other qualifies for the next round. Now we can easily predict the next numbers: 4, 2, and 1. Since there will be one champion, the sequence will end at 1, that is, the sequence has a finite number of terms. Sequences may be finite in number or infinite.

Look at the following sequence:

$$1000, 1100, 1210, \ldots$$

This is the total money owned by an investor at the end of each successive year. The capital increases by 10% every year. You can predict the next number in the sequence to be 1331. Each successive term here is 110% of, or 1.1 times, the previous term.

Can you recognize the pattern?

Real number sequences may follow an easily recognizable pattern or they may not. Recently a great deal of mathematical work has concentrated on deciding whether certain number sequences follow a pattern (that is, we can predict consecutive terms) or whether they are random (that is, we cannot predict consecutive terms). This work forms the basis of chaos theory, speech recognition, weather prediction and financial management, which are just a few examples of an almost endless list. In this book we will consider real number sequences which follow a pattern.

SEQUENCES

Definition

By the set of natural numbers we mean all positive integers and denote this set by ℕ. That is, ℕ = {1, 2, 3, ...}.

If someone asked you to list the squares of all the natural numbers, you might begin by writing

$$1, 4, 9, 16, 25, 36, ...$$

But you would soon realize that it is actually impossible to list all these numbers since there are an infinite number of them. However, we can represent this collection of numbers in several different ways.

A function is a relation between two sets A and B that assigns to each element of set A exactly one element of set B.

For example, we can also express the above list of numbers by writing

$$f(1), f(2), f(3), f(4), f(5), f(6), ..., f(n), ...$$

where $f(n) = n^2$. Here $f(1)$ is the first term, $f(2)$ is the second term, and so on. $f(n) = n^2$ is a **function** of n, defined in the set of natural numbers.

Definition

sequence

A function which is defined in the set of natural numbers is called a **sequence**.

However, we do not usually use functional notation to describe sequences. Instead, we denote the first term by a_1, the second term by a_2, and so on. So for the above list

$$a_1 = 1, a_2 = 4, a_3 = 9, a_4 = 16, a_5 = 25, a_6 = 36, ..., a_n = n^2, ...$$

Here, a_1 is the first term,

a_2 is the second term,

a_3 is the third term,

.
.
.

a_n is the nth term, or the **general term**.

Since this is just a matter of notation, we can use another letter instead of the letter a. For example, we can also use b_n, c_n, d_n, etc. as the name for the general term of a sequence.

Notation

We denote a sequence by (a_n), where a_n is written inside brackets. We write the general term of a sequence as a_n, where a_n is written without brackets. For the above example, if we write the general term, we write $a_n = n^2$.

If we want to list the terms, we write $(a_n) = (1, 4, 9, 16, ..., n^2, ...)$.

Sometimes we can also use a shorthand way to write a sequence:

$(a_n) = (n^2 + 4n + 1)$ means the sequence (a_n) with general term $a_n = n^2 + 4n + 1$.

Note

An expression like $a_{2.6}$ is nonsense since we cannot talk about the 2.6th term of a sequence. Remember that a sequence is a function which is defined in the set of natural numbers, and 2.6 is not a natural number. Clearly, expressions like a_0, a_{-1} are also meaningless. We say that such terms are **undefined**.

Note

In a sequence, n should always be a natural number, but the value of a_n may be any real number depending on the formula for the general term of the sequence.

Example 1 Write the first five terms of the sequence with general term $a_n = \frac{1}{n}$.

Solution Since we are looking for the first five terms, we just recalculate the general term for $n = 1, 2, 3, 4, 5$, which gives $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$.

Example 2 Given the sequence with general term $a_n = \frac{4n-5}{2n}$, find a_5, a_{-2}, a_{100}.

Solution We just have to recalculate the formula for a_n choosing instead of n the numbers 5, –2, and 100. So $a_5 = \frac{3}{2}$, and $a_{100} = \frac{395}{200} = \frac{79}{40}$. Clearly, a_{-2} is undefined, since –2 is not a natural number.

Example 3 Find a suitable general term b_n for the sequence whose first four terms are $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$.

Solution We need to find a pattern. Notice that the numerator of each fraction is equal to the term position and the denominator is one more than the term position, so we can write $b_n = \frac{n}{n+1}$.

Check Yourself

1. Write the first five terms of the sequence whose general term is $c_n = (-1)^n$.
2. Find a suitable general term a_n for the sequence whose first four terms are 2, 4, 6, 8.
3. Given the sequence with general term $b_n = 2n + 3$, find b_5, b_0, and b_{43}.

Answers

1. –1, 1, –1, 1, –1 2. $2n$ 3. 13, undefined, 89

Criteria for the Existence of a Sequence

If there is at least one natural number which makes the general term of a sequence undefined, then there is no such sequence.

Example 4 Is $a_n = \dfrac{2n+1}{n-2}$ a general term of a sequence? Why?

Solution No, because we cannot find a proper value for $n = 2$.

Example 5 Is $a_n = \sqrt{\dfrac{4-n}{2n+1}}$ a general term of a sequence? Why?

Solution Note that the expression $\sqrt{x}$ is only meaningful when $x \geq 0$. So we need $\dfrac{4-n}{2n+1} \geq 0$ to be true for any natural number n. If we solve this equation for n, the solution set is $(-\frac{1}{2}, 4]$, i.e. n is between $-\frac{1}{2}$ and 4, inclusive. When we take the natural numbers in this solution set, we get $\{1, 2, 3, 4\}$, which means that only a_1, a_2, a_3, a_4 are defined. So a_n is not the general term of a sequence.

Example 6 Is $a_n = \dfrac{n+1}{2n-1}$ a general term of a sequence? If yes, find $a_1 + a_2 + a_3$.

Solution $\dfrac{n+1}{2n-1}$ is not meaningful only when $n = \frac{1}{2} \notin \mathbb{N}$. Since a_n is defined for any natural number, it is the general term of a sequence. Choosing $n = 1, 2, 3$ we get $a_1 = 2$, $a_2 = 1$, $a_3 = 0.8$. So $a_1 + a_2 + a_3 = 3.8$.

Example 7 Given $b_n = 2n + 5$, find the term of the sequence (b_n) which is equal to

a. 25 b. 17 c. 96

Solution

a. $b_n = 25$
$2n + 5 = 25$
$n = 10$
10th term

b. $b_n = 17$
$2n + 5 = 17$
$n = 6$
6th term

c. $b_n = 96$
$2n + 5 = 96$
$n = 45.5 \notin \mathbb{N}$
not a term

THE FIBONACCI SEQUENCE AND THE GOLDEN RATIO

The sequence in the previous example is called the **Fibonacci sequence**, named after the 13th century Italian mathematician Fibonacci, who used it to solve a problem about the breeding of rabbits. Fibonacci considered the following problem:

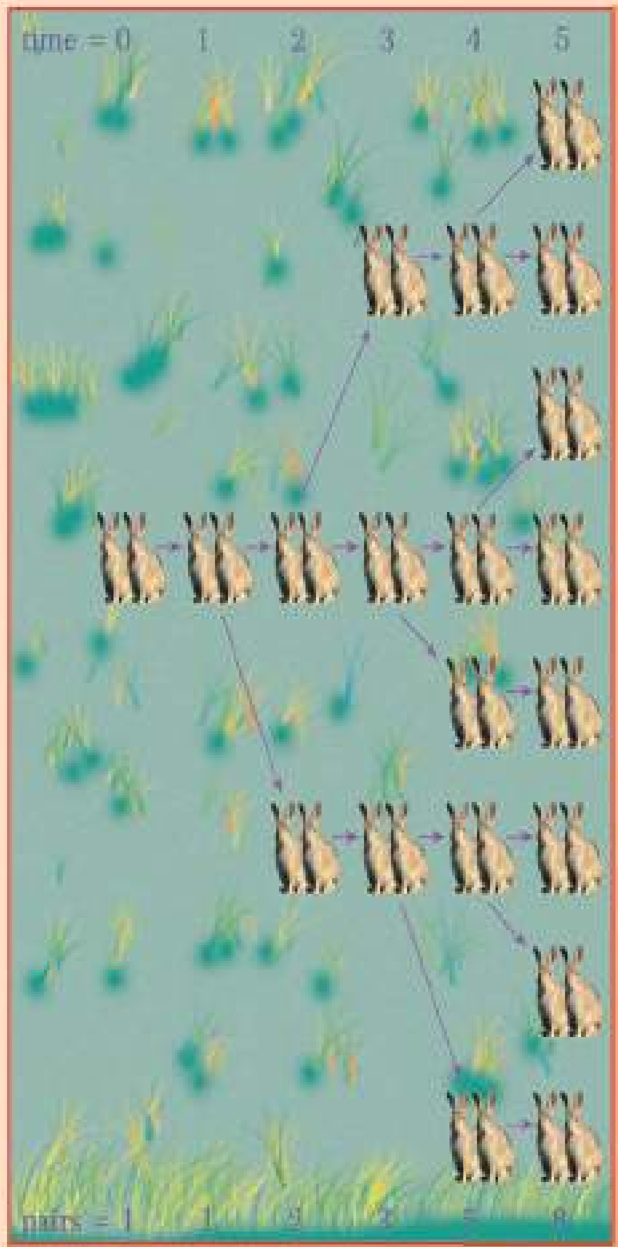
Suppose that rabbits live forever and that every month each pair produces a new pair that becomes productive at age two months. If we start with one newborn pair, how many pairs of rabbits will we have in the *n*th month?

As a solution, Fibonacci found the following sequence:

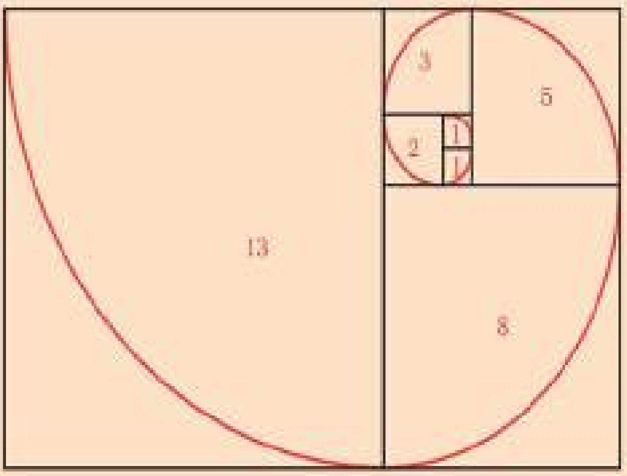
1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ...

This sequence also occurs in numerous other aspects of the natural world.

The planets in our solar system are spaced in a Fibonacci sequence.



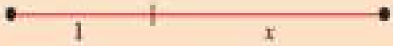
We can make a picture showing the Fibonacci numbers if we start with two small squares whose sides are each one unit long next to each other. Then we draw a square with side length two units (1 + 1 units) next to both of these. We can now draw a new square which touches the square with side one unit and the square with side two units, and therefore has side three units. Then we draw another square touching the two previous squares (side five units), and so on. We can continue adding squares around the picture, each new square having a side which is as long as the sum of the sides of the two previous squares. Now we can draw a spiral by connecting the quarter circles in each square, as shown on the next page. This is a spiral (the **Fibonacci Spiral**). A similar curve to this occurs in nature as the shape of a nautilus.



A nautilus has the same shape as the Fibonacci spiral.

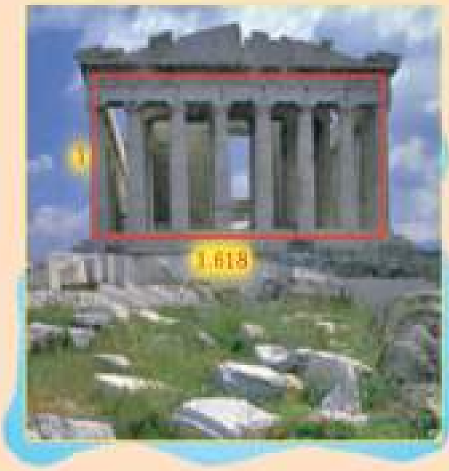
The ratio of two successive Fibonacci numbers $\frac{f_{n+1}}{f_n}$ gets closer to the number $\frac{1+\sqrt{5}}{2} \approx 1.618$ as the value of n gets bigger. This number is a special number in mathematics and is known as the **golden ratio**.

The ancient Greeks also considered a line segment divided into two parts such that the ratio of the shorter part of length one unit to the longer part is the same as the ratio of the longer part to the whole segment.

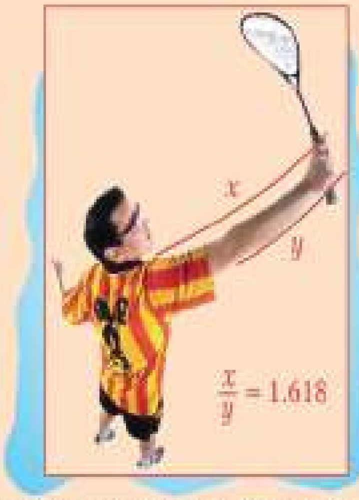


This leads to the equation $\frac{1}{x} = \frac{x}{1+x}$ whose positive solution is $x = \frac{1+\sqrt{5}}{2}$. Thus, the segment shown is divided into the golden ratio!

A rectangle in which the ratio of one side to the other gives the golden ratio is called a **golden rectangle**. The Golden Rectangle is a unique and a very important shape in mathematics. It appears in nature and music, and is also often used in art and architecture. The Golden Rectangle is believed to be one of the most pleasing and beautiful shapes for the human eye.



The golden ratio is frequently used in architecture.



The ratio of the length of your arm to the length from the elbow down to the end of your hand is approximately equal to the golden ratio.

EXERCISES 1

Sequences

1. State whether each term is a general term of a sequence or not.

   a. $3n - 76$ b. $\dfrac{n}{n+2}$ c. $\dfrac{2n+1}{2n-1}$

   d. $\dfrac{4}{n^2-4}$ e. $\dfrac{13}{4}$ f. $(-1)^n \dfrac{1}{n^3}$

   g. $\sqrt{n-5}$ h. $\sqrt{n^2+2n}$ i. $\sqrt{\dfrac{n^2-n-2}{n-2}}$

2. Find a suitable formula for the general terms of the sequences whose first few terms are given.

   a. 1, 3, 5 b. –1, 3, –5

   c. 0, 3, 8, 15 d. $-\dfrac{1}{5}, -\dfrac{8}{7}, -\dfrac{27}{9}$

   e. 2, 6, 12, 20, 30

3. Find the stated terms for the sequence with the given general term.

   a. $a_n = 2n + 3$, find the first three terms and a_{37}

   b. $a_n = \dfrac{3n+1}{n+7}$, find the first three terms and a_{33}

   c. $a_n = \sqrt{n^2+6n}$, find the first three terms and a_6

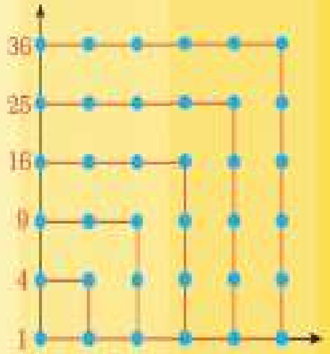
4. How many terms of the sequence with general term $a_n = n^2 - 6n - 16$ are negative?

5. How many terms of the sequence with general term $a_n = \dfrac{3n-7}{3n+5}$ are less than $\dfrac{1}{5}$?

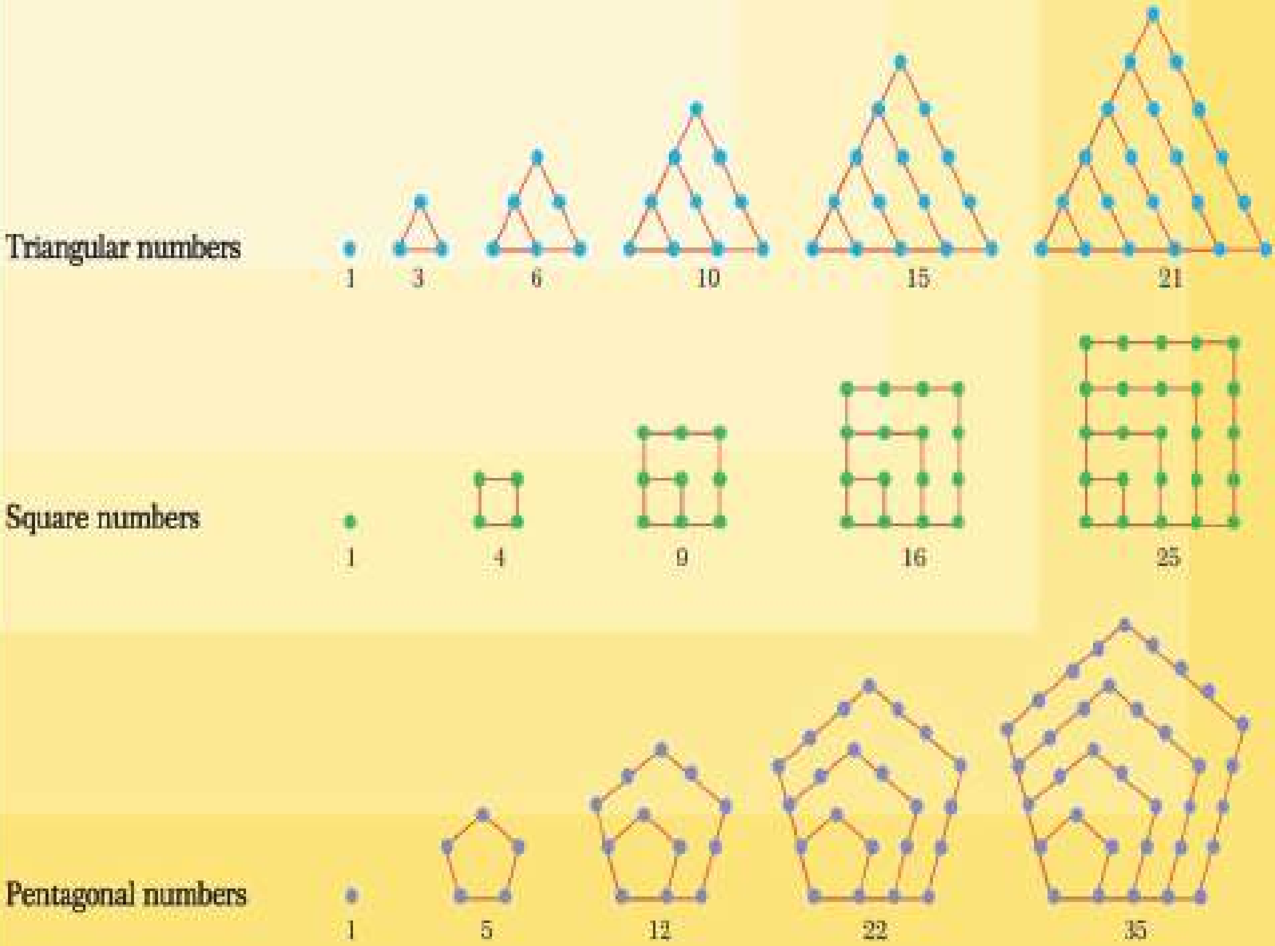
POLYGONAL NUMBERS

At the beginning of this book we looked at the sequence 1, 4, 9, 16, 25, 36, … . We call the numbers in this sequence **square numbers**. We can generate the square numbers by creating a sequence of nested squares like the one on the right. Starting from a common vertex, each square has sides one unit longer than the previous square. When we count the number of points in each successive square, we get the sequence of square numbers

(first square = 1 point, second square = 4 points, third square = 9 points, etc.).



Polygonal numbers are numbers which form sequences like the one above for different polygons. The Pythagoreans named these numbers after the polygons that defined them.



Polygonal numbers have many interesting relationships between them. For example, the sum of any two consecutive triangular numbers is a square number, and eight times any triangular number plus one is always a square number.

Can you find any more patterns? Can you find the general term for each set of polygonal numbers?

2 ARITHMETIC SEQUENCES

ARITHMETIC SEQUENCES

1. Definition

Let's look at the sequence 6, 10, 14, 18, ...

Obviously the difference between each term is equal to 4 and the sequence can be written as $a_{n+1} = a_n + 4$ where $a_1 = 6$.

For the sequence 23, 21, 19, ... the formula will be $a_{n+1} = a_n - 2$ where $a_1 = 23$.

In these examples, the difference between consecutive terms in each sequence is the same. We call sequences with this special property **arithmetic sequences**.

Definition **arithmetic sequence**

If a sequence (a_n) has the same difference d between its consecutive terms, then it is called an **arithmetic sequence**.

In other words, (a_n) is arithmetic if $a_{n+1} = a_n + d$ such that $n \in \mathbb{N}$, $d \in \mathbb{R}$. We call d the **common difference** of the arithmetic sequence. In this book, from now on we will use a_n to denote general term of an arithmetic sequence and d (the first letter of the Latin word *differentia*, meaning difference) for the common difference.

If d is positive, we say the arithmetic sequence is **increasing**. If d is negative, we say the arithmetic sequence is **decreasing**. What can you say when d is zero?

EXAMPLE 8 State whether the following sequences are arithmetic or not. If a sequence is arithmetic, find the common difference.

a. 7, 10, 13, 16, ... b. 3, −2, −7, −12, ... c. 1, 4, 9, 16, ... d. 6, 6, 6, 6, ...

Solution a. arithmetic, $d = 3$ b. arithmetic, $d = -5$ c. not arithmetic d. arithmetic, $d = 0$

EXAMPLE 9 State whether the sequences with the following general terms are arithmetic or not. If a sequence is arithmetic, find the common difference.

a. $a_n = 4n - 3$ b. $a_n = 2^n$ c. $a_n = n^2 - n$ d. $a_n = \dfrac{n^2 + 5n - 4}{n + 4}$

Solution a. $a_{n+1} = 4(n + 1) - 3 = 4n + 1$, so the difference between each consecutive term is $a_{n+1} - a_n = (4n + 1) - (4n - 3) = 4$, which is constant. Therefore, (a_n) is an arithmetic sequence and $d = 4$.

b. $a_{n+1} = 2^{n+1}$, so the difference between each consecutive term is $a_{n+1} - a_n = 2^{n+1} - 2^n = 2^n$, which is not constant. Therefore, (a_n) is not an arithmetic sequence.

c. $a_{n+1} = (n + 1)^2 - (n - 1)$, so the difference between two consecutive terms is $a_{n+1} - a_n = [(n + 1)^2 - (n + 1)] - (n^2 - n) = 2n$, which is not constant. Therefore, (a_n) is not an arithmetic sequence.

d. By rewriting the general term we have $a_n = \frac{(n+4)(n+1)}{n+4}$. Since $n \neq -4$ (since we are talking about a sequence), we have $a_n = n + 1$. Therefore, $a_{n+1} = (n + 1) + 1$, and the difference between the consecutive terms is $a_{n+1} - a_n = 1$, which is constant. Therefore, (a_n) is an arithmetic sequence and $d = 1$.

With the help of the above example we can notice that if the formula for general term of a sequence gives us a linear function, then it is arithmetic.

Note

The general term of an arithmetic sequence is linear.

2. General Term

Since arithmetic sequences have many applications, it is much better to express the general term directly, instead of recursively. The formula is derived as follows:

If (a_n) is arithmetic, then we only know that $a_{n+1} = a_n + d$. Let us write a few terms.

$a_1 = a_1$

$a_2 = a_1 + d$

$a_3 = a_2 + d = (a_1 + d) + d = a_1 + 2d$

$a_4 = a_3 + d = (a_1 + 2d) + d = a_1 + 3d$

$a_5 = a_1 + 4d$

$\vdots$

$a_n = a_1 + (n - 1)d$

This is the general term of an arithmetic sequence.

Arithmetic growth is linear.

GENERAL TERM FORMULA

The general term of an arithmetic sequence (a_n) with common difference d is

$$a_n = a_1+(n-1)d.$$

EXAMPLE 10 –3, 2, 7 are the first three terms of an arithmetic sequence (a_n). Find the twentieth term.

Solution We know that $a_1 = -3$ and $d = a_3 - a_2 = a_2 - a_1 = 5$. Using the general term formula,

$$a_n = a_1 + (n-1)d$$

$$a_{20} = -3 + (20-1) \cdot 5 = 92.$$

EXAMPLE 11 (a_n) is an arithmetic sequence with $a_1 = 4$, $a_8 = 25$. Find the common difference and a_{101}.

Solution Using the general term formula,

$$a_n = a_1 + (n-1)d$$

$$a_8 = a_1 + 7d$$

$25 = 4 + 7d$. So we have $d = 3$.

$$a_{101} = a_1 + (101-1)d = 4 + 100 \cdot 3 = 304$$

EXAMPLE 12 (a_n) is an arithmetic sequence with $a_1 = 3$ and common difference 4. Is 59 a term of this sequence?

Solution For 59 to be a term of the arithmetic sequence, it must satisfy the general term formula such that n is a natural number.

$$a_n = a_1 + (n-1)d$$

$$59 = 3 + (n-1) \cdot 4$$

$$59 = 4n - 1$$

$$n = 15$$

Since 15 is a natural number, 59 is the 15th term of this sequence.

EXAMPLE 13 Find the number of terms in the arithmetic sequence 1, 4, 7, ..., 91.

Solution Here we have a finite sequence. Using the general term formula,

$$a_n = a_1 + (n-1)d$$

$$91 = 1 + (n-1) \cdot 3$$

$$n = 31$$

Therefore, this sequence has 31 terms.

B. SUM OF THE TERMS OF AN ARITHMETIC SEQUENCE

1. Sum of the First *n* Terms

Let us consider an arithmetic sequence whose first few terms are 3, 7, 11, 15, 19.

The sum of the first term of this sequence is obviously 3. The sum of the first two terms is 10, the sum of the first three terms is 21, and so on. To write this in a more formal way, let us use S_n to denote **the sum of the first n terms**, i.e., $S_n = a_1 + a_2 + ... + a_n$. Now we can write:

$$S_1 = 3$$

$$S_2 = 3 + 7 = 10$$

$$S_3 = 3 + 7 + 11 = 21$$

$$S_4 = 3 + 7 + 11 + 15 = 36$$

$$S_5 = 3 + 7 + 11 + 15 + 19 = 55.$$

EXAMPLE 14 Given the arithmetic sequence with general term $a_n = 3n + 1$, find the sum of first three terms.

Solution $S_3 = a_1 + a_2 + a_3 = 4 + 7 + 10 = 21.$

How could we find S_{100} in the above example? Calculating terms and finding their sums takes time and effort for large sums. Since arithmetic sequences are of special interest and importance, we need a more efficient way of calculating the sums of arithmetic sequences. The following theorem meets our needs:

Theorem

The sum of the of first n terms of an arithmetic sequence (a_n) is $S_n = \frac{a_1 + a_n}{2} n$.

Proof $S_n = a_1 + a_2 + ... + a_{n-1} + a_n$ or

$S_n = a_n + a_{n-1} + ... + a_2 + a_1$.

Adding these equations side by side,

$$2S_n = (a_1 + a_n) + (a_2 + a_{n-1}) + ... + (a_{n-1} + a_2) + (a_n + a_1)$$

$$2S_n = (a_1 + a_n) + (a_1 + d + a_n - d) + ... + (a_n - d + a_1 + d) + (a_n + a_1)$$

$$2S_n = \underbrace{(a_1 + a_n) + (a_1 + a_n) + ... + (a_1 + a_n) + (a_1 + a_n)}_{n \text{ times}}$$

$$2S_n = (a_1 + a_n) \cdot n$$

$$S_n = \frac{a_1 + a_n}{2} \cdot n.$$

EXAMPLE 15 Given an arithmetic sequence with $a_1 = 2$ and $a_6 = 17$, find S_6.

Solution Using the sum formula,

$$S_6 = \frac{a_1 + a_6}{2} \; 6 = (2 + 17) \; 3 = 57.$$

EXAMPLE 16 Given an arithmetic sequence with $a_1 = -14$ and $d = 5$, find S_{27}.

Solution Using the sum formula,

$S_{27} = \frac{a_1 + a_{27}}{2} \cdot 27$ requires $a_{27} = a_1 + 26d = -14 + 26 \cdot 5 = 116$.

Therefore, $S_{27} = \frac{-14 + 116}{2} \cdot 27 = 1377$.

EXAMPLE 17 Given an arithmetic sequence with $a_1 = 56$ and $a_{11} = -14$, find S_{15}.

Solution Using the sum formula,

$S_{15} = \frac{a_1 + a_{15}}{2} \cdot 15$, so we need to find a_{15}. Let us calculate using a_{11}:

$a_{11} = a_1 + 10d$

$-14 = 56 + 10d$, so $d = -7$ and

$a_{15} = a_1 + 14d = 56 + 14 \cdot (-7) = -42$.

Therefore, $S_{15} = \frac{56 - 42}{2} \cdot 15 = 105$.

EXAMPLE 18 If $-5 + ... + 49 = 616$ is the sum of the terms of a finite arithmetic sequence, how many terms are there in the sequence?

Solution Let us convert the problem into algebraic language:

$a_1 = -5$, $a_p = 49$, and $S_p = 616$, and we need to find p.

Using the sum formula,

$S_p = \frac{a_1 + a_p}{2} \cdot p$, that is, $616 = \frac{-5 + 49}{2} \cdot p$, so $p = 28$. So 28 numbers were added.

Since $a_n = a_1 + (n - 1)d$, we can also rewrite the sum formula as follows:

Check Yourself

1. Given an arithmetic sequence with $a_1 = 4$ and $a_{10} = 15$, find S_{10}.
2. Given an arithmetic sequence with $a_{13} = 26$ and $d = -2$, find S_{13}.
3. Given an arithmetic sequence with $a_1 = 9$ and $S_8 = 121$, find d.
4. Find the sum of all the multiples of 3 between 20 and 50.

Answers

1. 95 2. 494 3. 1.75 4. 345

EXERCISES 2

A. Arithmetic Sequences

1. State whether the following sequences are arithmetic or not.

   a. $(a_n) = (n^2)$ b. $(\sqrt{2}, \sqrt{2}, \sqrt{2}, \ldots)$ c. $(a_n) = (4n+7)$

2. Find the formula for the general term a_n of the arithmetic sequence with the given common difference and first term.

   a. $d = 2, a_1 = 3$ b. $d = \sqrt{3}, a_1 = 1$

   c. $d = 0, a_1 = 0$ d. $d = -\frac{3}{2}, a_1 = -3$

   e. $d = -1, a_1 = 0$ f. $d = 7, a_1 = \sqrt{2}$

   g. $d = b + 3, a_1 = 2b + 7$

3. Find the common difference and the general term a_n of the arithmetic sequence with the given terms.

   a. $a_1 = 3, a_2 = 5$ b. $a_1 = 4, a_4 = 10$

   c. $a_3 = \sqrt{2}, a_8 = 6\sqrt{2}$ d. $a_{12} = -12, a_{24} = -24$

   e. $a_5 = 8, a_{17} = 8$ f. $a_6 = 6, a_{20} = -34$

   g. $a_3 = 1, a_5 = 2$

   h. $a_2 = 2x - y, a_8 = x + 2y$

4. Find the general term of the arithmetic sequence using the given data.

   a. $a_{n+1} = a_n + 7, a_1 = -2$

   b. $a_{17} = 41, d = 4$

5. Fill in the blanks to form an arithmetic sequence.

   a. __, __, __, 3, __, __, __, 32.

   b. 13, $\underbrace{\_\_, \_\_, \ldots, \_\_}_{\text{seven terms}}$, 45

6. In an arithmetic sequence the first term is –1 and the common difference is 3. Is 27 a term of this sequence?

7. Given that the following sequences are arithmetic, find the missing value.

a. $\frac{a_{12}+a_{20}}{2}=?$ b. $a_6=\frac{a_4+?}{2}$

B. Sum of the Terms of an Arithmetic Sequence

8. For each arithmetic sequence (a_n) find the missing value.

a. $a_1=-5,\ a_8=18,\ S_8=?$

b. $a_1=-3,\ a_7=27,\ S_{40}=?$

c. $a_1=7,\ S_{16}=332,\ d=?$

d. $d=\frac{5}{3},\ S_{34}=1173,\ a_1=?$

e. $a_1=2,\ a_{n+1}=a_n-2,\ S_{23}=?$

f. $a_1=\frac{3}{2},\ d=\frac{1}{2},\ S_p=1700,\ p=?$

g. $S_{100}=10000,\ a_{100}=199,\ a_{10}=?$

h. $a_n=-5n-10,\ S_7=?$

i. $a_1=5,\ a_p=20,\ S_p=250,\ p=?$

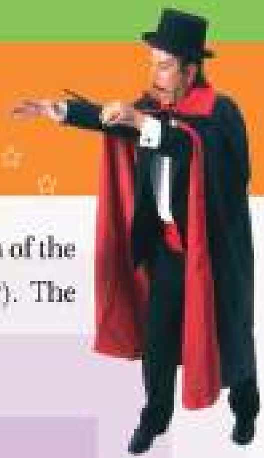
j. $S_{60}=3840,\ a_1=5,\ a_{61}=?$

k. $a_1=3,\ a_{10}-a_7=-6,\ S_{20}=?$

l. $a_1=1,\ S_{22}-S_{18}=238,\ a_7=?$

m. $d=5,\ S_{16}-S_{10}=308,\ a_1=?$

Magic Squares

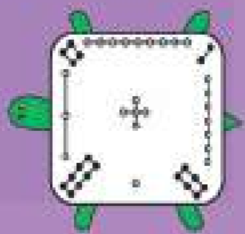


A **magic square** is an arrangement of natural numbers in a square matrix so that the sum of the numbers in each column, row, and diagonal is the same number (the **magic number**). The number of cells on one side of the square is called the **order** of the magic square.

Here is one of the earliest known magic squares:

4	9	2
3	5	7
8	1	6

It is a third order magic square constructed by using the numbers 1, 2, 3, ..., 9. Notice that the numbers in each row, column, and diagonal add up to the number 15, and 1, 2, 3, ...,9 form an arithmetic sequence. This magic square was possibly constructed in 2200 B.C. in China. It is known as the **Lo-Shu** magic square.



The famous Lo-Shu is the oldest known magic square in the world. According to the legend, the figure above was found on the back of a turtle which came from the river Lo. The word 'Shu' means 'book', so 'Lo-Shu' means 'The book of the river Lo'.

Below is another magic square, this time of order four. Note that its elements are from the finite arithmetic sequence 7, 10, 13, 16, ..., 52, and the magic number is 118.

52	13	10	43
19	34	37	28
31	22	25	40
16	49	46	7

What kind of relation exists between the sequence and the magic number? Given any finite arithmetic sequence of n^2 terms is it always possible to construct a magic square? If the numbers do not form an arithmetic sequence, is it possible to construct a magic square?

Try constructing your own magic square of order three using the numbers 4,8,12, ...,36.

There are many unsolved puzzles concerning magic squares. The puzzle of Yang-Hui, which was solved in the year 2000, was one of them. According to the legend the 13th century Chinese mathematician Yang-Hui gave the emperor Sung his last magic square as a gift. This is Yang-Hui's square:

1668	198	1248
618	1038	1458
828	1878	408

+1

1669	199	1249
619	1039	1459
829	1879	409

The special property of Yang-Hui's square was that the square had elements of a finite arithmetic sequence with common difference 210 such that when 1 was added to each cell it would become another magic square with all elements prime numbers. But the emperor wanted the magic square to also give prime numbers when 1 was subtracted from each cell. He promised some land along the river to the mathematician if it was completed. Unfortunately, the life of Yang-Hui wasn't long enough to solve this puzzle. Below is the solution to the problem, calculated 725 years later:

372839669	241608569	267854789
189116129	294101009	399085889
320347229	346593449	215362349

-1

372839670	241608570	267854790
189116130	294101010	399085890
320347230	346593450	215362350

+1

372839671	241608571	267854791
189116131	294101011	399085891
320347231	346593451	215362351

3 GEOMETRIC SEQUENCES

A. GEOMETRIC SEQUENCES

1. Definition

In the previous section, we learned about arithmetic sequences, i.e. sequences whose consecutive terms have a common difference. In this chapter we will look at another type of sequence, called a **geometric sequence**. Geometric sequences play an important role in mathematics.

A sequence is called geometric if the ratio between each consecutive term is common. For example, look at the sequence 3, 6, 12, 24, 48, …

Obviously the ratio of each term to the previous term is equal to 2, so we can formulize the sequence as $b_{n+1} = b_n \cdot 2$. The consecutive terms of the sequence have a common ratio (2), so this sequence is geometric.

For the sequence 625, 125, 25, 5, 1, … the formula will be $b_{n+1} = b_n \cdot \frac{1}{5}$. The common ratio in this sequence is $\frac{1}{5}$.

Definition

geometric sequence

If a sequence (b_n) has the same ratio q between its consecutive terms, then it is called a **geometric sequence.**

In other words, (b_n) is geometric if $b_{n+1} = b_n \cdot q$ such that $n \in \mathbb{N}$, $q \in \mathbb{R}$. q is called the **common ratio** of the sequence. In this book, from now on we will use b_n to denote the general term of a geometric sequence, and q to denote the common ratio.

If $q > 1$, the geometric sequence is **increasing** when $b_1 > 0$ and **decreasing** when $b_1 < 0$.

If $0 < q < 1$, geometric sequence is **increasing** when $b_1 < 0$ and **decreasing** when $b_1 > 0$.

If $q < 0$, then the sequence is **not monotone**.

What can you say if $q = 1$? What about $q = 0$?

EXAMPLE 19 State whether the following sequences are geometric or not. If a sequence is geometric, find the common ratio.

a. 1, 2, 4, 8, … b. 3, 3, 3, 3, … c. 1, 4, 9, 16, … d. $5, -1, \frac{1}{5}, -\frac{1}{25}, \ldots$

Solution a. geometric, $q = 2$ b. geometric, $q = 1$ c. not geometric d. geometric, $q = -\frac{1}{5}$

EXAMPLE 20 State whether the sequences with the given general terms are geometric or not. If a sequence is geometric, find the common ratio.

a. $b_n = 3^n$ b. $b_n = n^2 + 3$ c. $b_n = 3 \cdot 2^{n+3}$ d. $b_n = 3n + 5$

Solution a. $b_{n+1} = 3^{n+1}$, so the ratio between each consecutive term is $\frac{b_{n+1}}{b_n} = \frac{3^{n+1}}{3^n} = 3$, which is constant. So (b_n) is a geometric sequence and $q = 3$.

b. $b_{n+1} = (n+1)^2 + 3$, so the ratio between each consecutive term is

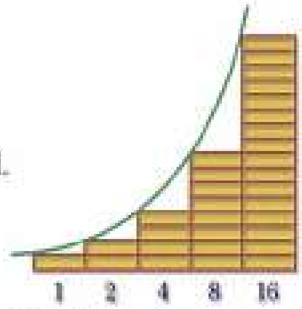
$\frac{b_{n+1}}{b_n} = \frac{(n+1)^2+3}{n^2+3} = \frac{n^2+2n+4}{n^2+3}$, which is not constant. So (b_n) is not a geometric sequence.

c. $b_{n+1} = 3 \cdot 2^{n+4}$, so the ratio between each consecutive term is $\frac{b_{n+1}}{b_n} = \frac{3 \cdot 2^{n+4}}{3 \cdot 2^{n+3}} = 2$, which is constant. So (b_n) is a geometric sequence and $q = 2$.

d. Since the general term has a linear form, this is an arithmetic sequence. It is not geometric.

With the help of the above example we can see that if the formula for the general term of a sequence gives us an exponential function with a linear exponent (a function with only one exponent variable), then it is geometric.

Note
The general term of a geometric sequence is exponential.



Geometric growth is exponential.

2. General Term

We have seen that for a geometric sequence, $b_{n+1} = b_n \cdot q$. This formula is defined recursively. If we want to make faster calculations, we need to express the general term of a geometric sequence more directly. The formula is derived as follows:

If (b_n) is geometric, then we only know that $b_{n+1} = b_n \cdot q$. Let us write a few terms.

$b_1 = b_1$

$b_2 = b_1 \cdot q$

$b_3 = b_2 \cdot q = (b_1 \cdot q) \cdot q = b_1 \cdot q^2$

$b_4 = b_3 \cdot q = (b_1 \cdot q^2) \cdot q = b_1 \cdot q^3$

$b_5 = b_1 \cdot q^4$

$\vdots$

$b_n = b_1 \cdot q^{n-1}$

This is the general term of a geometric sequence.

GENERAL TERM FORMULA

The general term of a geometric sequence (b_n) with common ratio q is

$$b_n = b_1 \cdot q^{n-1}$$

EXAMPLE 21 If 100, 50, 25 are the first three terms of a geometric sequence (b_n), find the sixth term.

Solution We can calculate the common ratio as $q = \frac{b_3}{b_2} = \frac{b_2}{b_1} = \frac{1}{2}$, so $b_1 = 100$, $q = \frac{1}{2}$.

Using the general term formula, $b_n = b_1 \cdot q^{n-1}$, so $b_6 = 100 \cdot (\frac{1}{2})^{6-1} = \frac{25}{8}$.

EXAMPLE 22 (b_n) is a geometric sequence with $b_1 = \frac{1}{3}$, $q = 3$. Find b_4.

Solution Using the general term formula,

$b_n = b_1 \cdot q^{n-1}$. Therefore, $b_4 = \frac{1}{3} \cdot 3^{4-1} = 9$.

EXAMPLE 23 (b_n) is a geometric sequence with $b_1 = -15$, $q = \frac{1}{5}$. Find the general term.

Solution Using the general term formula, $b_n = b_1 \cdot q^{n-1}$.

Therefore, $b_n = -15 \cdot \left(\frac{1}{5}\right)^{n-1} = -15 \cdot \left(\frac{1}{5}\right)^{n} \cdot \left(\frac{1}{5}\right)^{-1} = -75 \cdot \left(\frac{1}{5}\right)^{n}$.

How can you relate this building to a geometric sequence?

EXAMPLE 24 Consider the geometric sequence (b_n) with $b_1 = \frac{1}{9}$ and $q = 3$. Is 243 a term of this sequence?

Solution Using the general term formula,

$b_n = b_1 \cdot q^{n-1}$ and so $b_n = \frac{1}{9} \cdot 3^{n-1}$.

Now $243 = \frac{1}{9} \cdot \frac{3^n}{3}$, and so $3^n = 3^8$. Therefore, $n = 8$.

Since 8 is a natural number, 243 is the eighth term of this sequence.

EXAMPLE 25 In a monotone geometric sequence $b_1 \cdot b_5 = 12$, $\frac{b_2}{b_4} = 3$. Find b_2.

Solution $\frac{b_2}{b_4} = 3$, that is $\frac{b_1 \cdot q}{b_1 \cdot q^3} = 3$. So $q = \pm\frac{1}{\sqrt{3}}$.

Since the sequence is monotone, we take $q = \frac{1}{\sqrt{3}}$.

$b_1 \cdot b_5 = 12$, that is $b_1 \cdot b_1 \cdot q^4 = 12$.

$b_1^2 \cdot \frac{1}{9} = 12$, that is $b_1 = 6\sqrt{3}$. So $b_2 = b_1 \cdot q = 6\sqrt{3} \cdot \frac{1}{\sqrt{3}} = 6$.

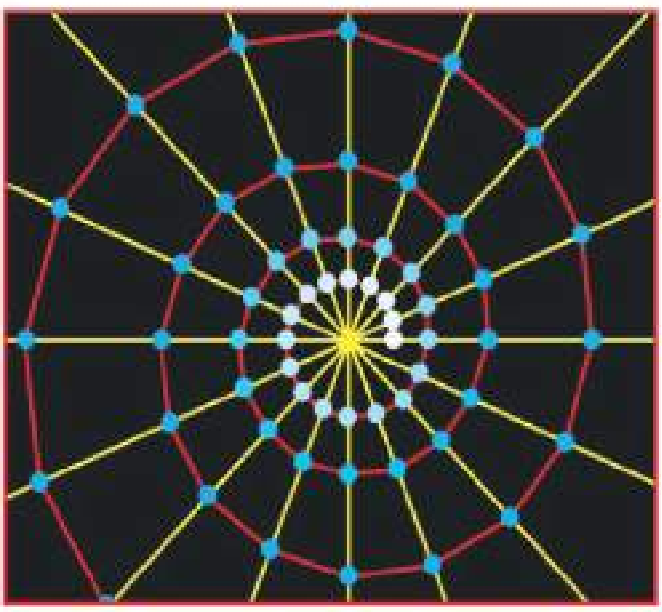
Why? Would the answer change if the sequence was not monotone? Why?

Check Yourself

1. Is the sequence with general term $b_n = \frac{1}{3} \cdot 4^{n+3}$ a geometric sequence? Why?
2. $\frac{3}{16}, \frac{3}{8}, \frac{3}{4}$ are the first three terms of a geometric sequence (b_n). Find the eighth term.
3. (b_n) is a non-monotone geometric sequence with $b_1 = \frac{1}{4}$, $b_7 = 16$. Find the common ratio of the sequence and b_4.
4. (b_n) with is a geometric sequence with $b_1 = -3$, $q = -2$. Is -96 a term of this sequence?

Answers

1. yes, because the general term formula is exponential 2. 24 3. $q = -2$; $b_4 = -2$ 4. no



B. SUM OF THE TERMS OF A GEOMETRIC SEQUENCE

1. Sum of the First *n* Terms

Let us consider the geometric sequence with first few terms 1, 2, 4, 8, 16.

The sum of the first term of this sequence is obviously 1. The sum of the first two terms is 3, the sum of the first three terms is 7, and so on. To write this in a more formal way, let us use S_n to denote **the sum of the first *n* terms**, i.e. $S_n = b_1 + b_2 + \ldots + b_n$. Now,

$$S_1 = 1$$
$$S_2 = 1 + 2 = 3$$
$$S_3 = 1 + 2 + 4 = 7$$
$$S_4 = 1 + 2 + 4 + 8 = 15$$
$$S_5 = 1 + 2 + 4 + 8 + 16 = 31.$$

EXAMPLE 26 Given the geometric sequence with general term $b_n = 3 \cdot (-2)^n$, find the sum of first three terms.

Solution $S_3 = b_1 + b_2 + b_3 = -6 + 12 - 24 = -18$

How could we find S_{100} in the previous example? Calculating terms and finding their sums takes time and effort for large sums. As geometric sequences grow very fast, we need a more efficient way of calculating these sums. The following theorem meets our needs:

Theorem

The sum of the first n terms of a geometric sequence (b_n) is $S_n = b_1 \cdot \frac{1-q^n}{1-q}$, $q \neq 1$.

Proof $S_n = b_1 + b_2 + b_3 + ... + b_{n-1} + b_n$

$S_n = b_1 + b_1 \cdot q + b_1 \cdot q^2 + ... + b_1 \cdot q^{n-2} + b_1 \cdot q^{n-1}$ (1)

$q \cdot S_n = b_1 \cdot q + b_1 \cdot q^2 + b_1 \cdot q^3 + ... + b_1 \cdot q^{n-1} + b_1 \cdot q^n$ (2)

Subtracting (2) from (1), we get

$S_n - q \cdot S_n = b_1 - b_1 \cdot q^n$

$S_n = b_1 \cdot \frac{1-q^n}{1-q}$

EXAMPLE 27 Given a geometric sequence with $b_1 = \frac{1}{81}$ and $q = 3$, find S_6.

Solution Using the sum formula,

$S_n = b_1 \cdot \frac{1-q^n}{1-q}$, so $S_6 = \frac{1}{81} \cdot \frac{1-3^6}{1-3} = \frac{364}{81}$.

EXAMPLE 28 Given a geometric sequence with $S_6 = 3640$ and $q = 3$, find b_1.

Solution Using the sum formula,

$S_6 = b_1 \cdot \frac{1-q^6}{1-q}$, so $3640 = b_1 \cdot \frac{1-3^6}{1-3}$, and so $b_1 = 10$.

EXAMPLE 29 Given a geometric sequence with $q = \frac{1}{3}$, $b_p = 5$ and $S_p = 1820$, find b_1.

Solution Using the sum formula,

$S_p = b_1 \cdot \frac{1-q^p}{1-q} = \frac{b_1 - b_1 \cdot q^p}{1-q} = \frac{b_1 - b_{p+1}}{1-q} = \frac{b_1 - b_p \cdot q}{1-q}$, so $1820 = \frac{b_1 - 5 \cdot \frac{1}{3}}{1 - \frac{1}{3}}$. Therefore, $b_1 = 1215$.

EXERCISES

A. Geometric Sequences

1. State whether the following sequences are geometric or not.

   a. $(2, -5, \frac{25}{2}, \ldots)$ b. $(b_n) = (4^{n^2-3})$

   c. $(b_n) = (2n + 7)$

2. Find the general term of the geometric sequence with the given qualities.

   a. $b_1 = 5, q = 2$ b. $b_1 = -3, q = \frac{1}{2}$

   c. $b_1 = 1000, q = \frac{1}{10}$ d. $b_1 = \sqrt{3}, q = \sqrt{3}$

   e. $b_1 = 4, b_4 = 32$ f. $b_1 = 3, b_5 = \frac{1}{27}$

   g. $b_3 = 32, b_6 = \frac{1}{2}$ h. $b_5 = 5, b_{25} = 5$

   i. $b_1 = 2, b_6 = 8\sqrt{2}$

3. Fill in the blanks to form a geometric sequence.

   a. $3 - 2\sqrt{2}$, __, $3 + 2\sqrt{2}$

   b. __, __, 36, __, 4

4. Find the general term of the geometric sequence with $b_4 = b_2 + 24$ and $b_2 + b_3 = 6$.

5. Write the first four terms of the non-monotone geometric sequence that is formed by inserting nine terms between –3 and –729.

6. Given a geometric sequence with $b_6 = 4b_4$ and $b_3 \cdot b_6 = 1152$, find b_1.

B. Sum of the Terms of a Geometric Sequence

7. For each geometric sequence (b_n) find the missing value.

   a. $b_1 = -\frac{3}{2}, q = -2, S_7 = ?$

   b. $b_2 = 6, b_7 = 192, S_{11} = ?$

   c. $b_2 = 1, b_3 \cdot b_2 = 64 \cdot b_4 \cdot b_5, S_5 = ?$

   d. $S_3 = 111, q^3 = 4, S_6 = ?$

8. The general term of a geometric sequence is $b_n = 3 \cdot 2^n$. Find S_{10}.

1. Which terms can be the general term of a sequence?

I. $\frac{n}{n-2}$ II. 3 III. $n^2 + 2n + 3$

IV. $\sqrt{7-n}$ V. 3^n VI. n^n

A) I, II, III, IV B) II, III, IV, VI
C) I, II, III, VI D) II, III, V, VI
E) III, IV, V, VI

2. Which of the following can be the general term of the sequence with the first four terms 3, 5, 7, 9?

A) $2n - 1$ B) $2n$ C) $2n + 1$
D) $n + 1$ E) $n^2 + 2$

3. Given $a_1 = 2$, and $a_{n+1} = \frac{2a_n + 5}{2}$ for $n \geq 1$, find a_{11}.

A) 27 B) 25 C) 22
D) $\frac{27}{2}$ E) $\frac{25}{2}$

4. How many terms of the sequence with general term $\frac{n^2 - 2n + 36}{n}$ are natural numbers?

A) 5 B) 6 C) 7 D) 8 E) 9

5. How many terms of the sequence with general term $a_n = (\frac{2n+1}{n+9})$ are less than $\frac{1}{3}$?

A) 0 B) 1 C) 2 D) 3 E) 4

6. Given $a_n = (\frac{3n^2 - 5n}{n+k-3})$ and $a_5 = 3$, find k.

A) 3 B) 5 C) $\frac{22}{3}$
D) $\frac{35}{3}$ E) $\frac{44}{3}$

7. How many of the following sequences are decreasing?

I. $(a_n) = (\frac{3n-5}{n+2})$ II. $(b_n) = (n-3)^2$

III. $(c_n) = (-1^n)$ IV. $(d_n) = (\frac{1}{n+1})$

V. $(e_n) = (\frac{(-1)^n}{3n+2})$

A) 1 B) 2 C) 3 D) 4 E) 5

8. What is the minimum value in the sequence formed by $a_n = (\frac{2n+3}{3n-7})$?

A) –1 B) –3 C) –2 D) –7 E) –8

9. Which one of the following is the general term of an arithmetic sequence?

A) $n^2 + 2n$ B) $4n + 5$ C) n^3 D) $2^n + 3$ E) 5^n

10. If $\frac{1}{3}$, a, b, c, $\frac{5}{8}$ are consecutive terms of an arithmetic sequence, find $a + b + c$.

A) $\frac{7}{24}$ B) $\frac{23}{24}$ C) $\frac{21}{16}$ D) $\frac{23}{16}$ E) $\frac{69}{49}$

11. (a_n) is an arithmetic sequence with $a_{11} = 8$ and $a_{20} = 35$. Find a_3.

A) –3 B) –6 C) –16 D) –22 E) –28

12. (a_n) is arithmetic sequence with $a_1 = 7$ and common difference $\frac{1}{3}$. Find the general term.

A) $3n + 4$ B) $\frac{n+7}{3}$ C) $\frac{n-4}{3}$ D) $\frac{n+4}{3}$ E) $\frac{n+20}{3}$

13. (a_n) is an arithmetic sequence such that $a_3 + a_4 = 23$ and $a_5 + a_6 = 37$. Find a_8.

A) 49 B) 47 C) 45 D) 44 E) 43

14. (a_n) is a finite arithmetic sequence with first term $\frac{1}{2}$, last term $\frac{1}{16}$, and sum 9. How many terms are there in this sequence?

A) 9 B) 16 C) 32 D) 48 E) 64

15. $x - 2$, $x + 8$, $3x + 2$ form an arithmetic sequence. Find x.

A) 12 B) 11 C) 10 D) 9 E) 8

16. (a_n) is an arithmetic sequence with $S_4 = 3(S_8 - S_7)$ and $a_1 = 1$. Find the common difference.

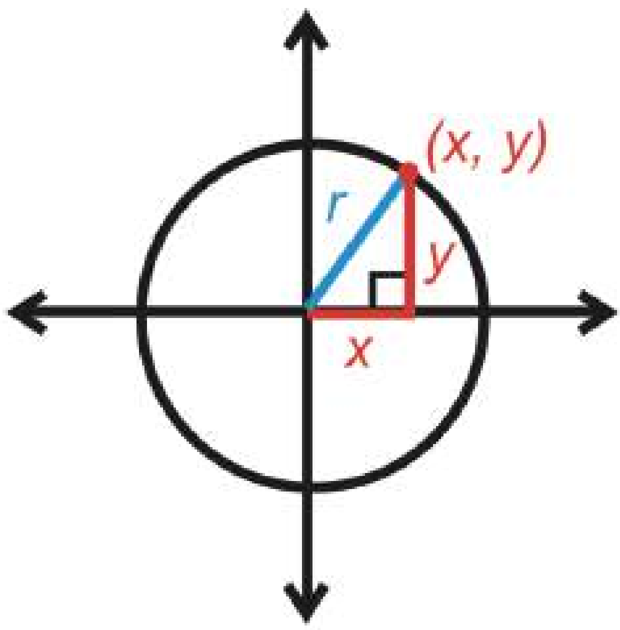
A) $-\frac{2}{51}$ B) $-\frac{13}{51}$ C) $\frac{2}{51}$ D) $\frac{13}{51}$ E) $\frac{15}{51}$

Chapter 3

CONIC SECTION

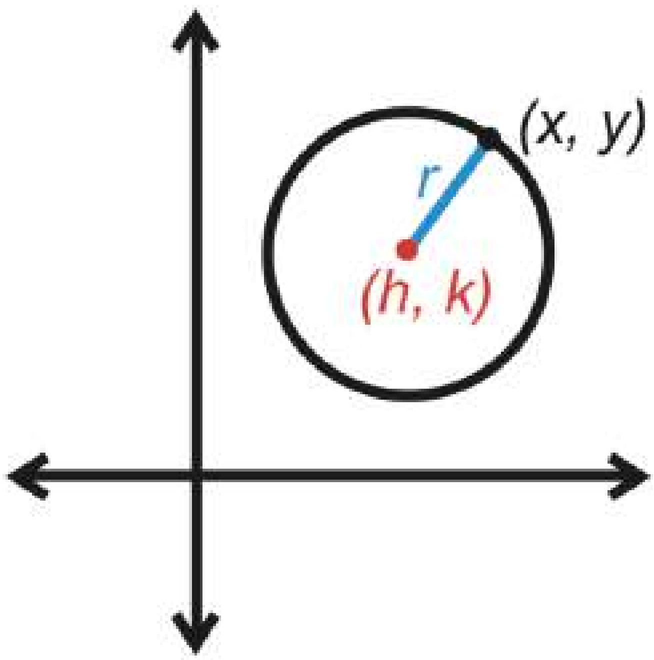
A-Circles in the Coordinate Plane

Recall that a circle is the set of all points in a plane that are the same distance from the center. This definition can be used to find an equation of a circle in the coordinate plane.



Let's start with the circle centered at (0, 0). If (x,y) is a point on the circle, then the distance from the center to this point would be the radius, r. x is the horizontal distance and y is the vertical distance. This forms a right triangle. From the Pythagorean Theorem, the equation of a circle *centered at the* ***origin*** is $x^2+y^2=r^2$.

The center does not always have to be on (0, 0). If it is not, then we label the center (h,k). We would then use the **Distance Formula** to find the length of the radius.

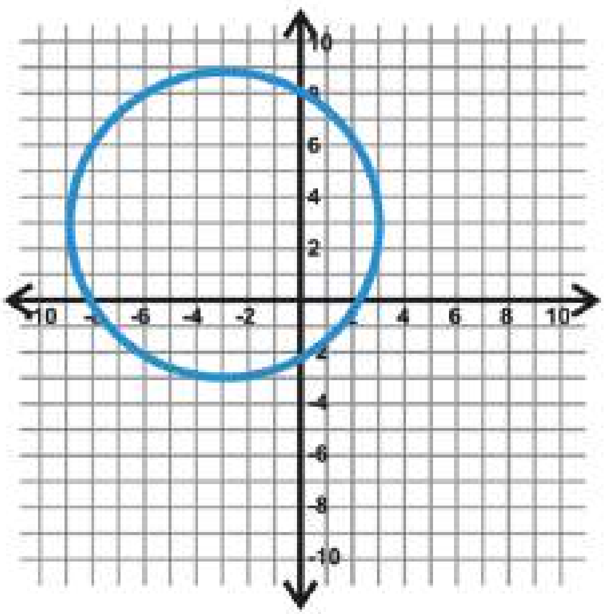


$$r = \sqrt{(x-h)^2 + (y-k)^2}$$

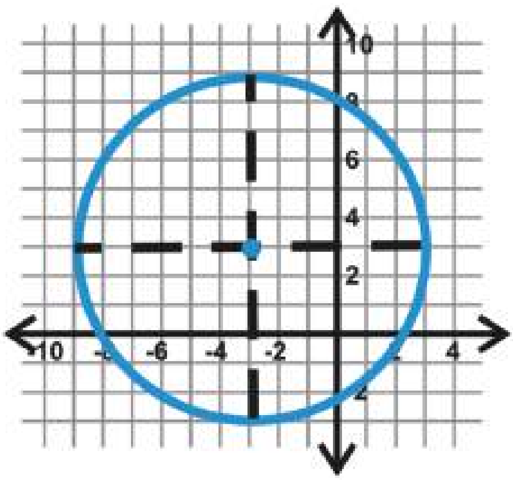
If you square both sides of this equation, then you would have the standard equation of a circle. **The standard equation of a circle with center** (h,k) **and radius** r **is** $r^2=(x-h)^2+(y-k)^2$**.**

B- Finding the Equation of a Circle

Find the equation of the circle below



First locate the center. Draw in the horizontal and vertical diameters to see where they intersect.



From this, we see that the center is (-3, 3). If we count the units from the center to the circle on either of these diameters, we find r=6. Plugging this into the equation of a circle, we get:
$(x-(-3))^2+(y-3)^2=6^2$ or $(x+3)^2+(y-3)^2=36$.

C- Determining if Points are on a Circle

Determine if the following points are on $(x+1)^2+(y-5)^2=50$.
Plug in the points for x and y in $(x+1)^2+(y-5)^2=50$

a) (8, -3)

$(8+1)^2+(-3-5)^2 = 9^2+(-8)^2 = 81+145 \neq 50$

(8,-3) is *not* on the circle.

b) (-2, -2)

$(-2+1)^2+(-2-5)^2=(-1)^2+(-7)^2=1+49=50$

(-2, -2) is on the circle

Examples

Find the center and radius of the following circles.

Example 1

$(x-3)^2+(y-1)^2=25$

Rewrite the equation as $(x-3)^2+(y-1)^2=5^2$.

The center is (3, 1) and r = 5.

Example 2

$(x+2)^2+(y-5)^2=49$

Rewrite the equation as $(x-(-2))^2+(y-5)^2=7^2$.

The center is (-2, 5) and r = 7.

Keep in mind that, due to the minus signs in the formula, the coordinates of the center have the ***opposite signs*** of what they may initially appear to be.

Example 3

Find the equation of the circle with center (4, -1) and which passes through (-1, 2).

First plug in the center to the standard equation.

$(x-4)^2+(y-(-1))^2 \Rightarrow (x-4)^2+(y+1)^2=r^2$

Now, plug in (-1, 2) for x and y and solve for r.

$(-1-4)^2+(2+1)^2 \Rightarrow (-5)^2+(3)^2 \Rightarrow 25+9=r^2 \Rightarrow 34$

Substituting in 34 for r^2, the equation is $(x-4)^2+(y+1)^2=34$

D- Convert a Circle Equation to the Standard Form

When the equation of a circle appears in the standard form, it provides you with all you need to know about the circle: its center and radius. With these two bits of information, you can sketch the graph of the circle.

The equation $x^2 + y^2 + 6x - 4y - 3 = 0$, for example, is the equation of a circle. You can change this equation to the standard form by completing the square for each of the variables. Just follow these steps:

1. Change the order of the terms so that the x 's and y's are grouped together and the constant appears on the other side of the equal sign.
2. Leave a space after the groupings for the numbers that you need to add:

   $x^2 + 6x + y^2 - 4y = 3$
3. Complete the square for each variable, adding the number that creates perfect square trinomials.
4. In the case of the x's, you add 9, and with the y's, you add 4. Don't forget to also add 9 and 4 to the right:

   $x^2 + 6x + 9 + y^2 - 4y + 4 = 3 + 9 + 4$

   When it's simplified, you have $x^2 + 6x + 9 + y^2 - 4y + 4 = 16$
5. Factor each perfect square trinomial. The standard form for the equation of this circle is: $(x + 3)^2 + (y - 2)^2 = 16$.

1-Find the equation of the circle.

All you need for the equation of a circle is its center (you know it) and its radius. The radius of the circle is just the distance from its center to any point on the circle. Since the point of tangency is given, that's the point to use.

$$\text{Distance}_{C \text{ to } T} = \sqrt{(4-1)^2 + (6-2)^2}$$
$$= \sqrt{3^2 + 4^2}$$
$$= 5$$

Now you finish by plugging the center coordinates and the radius into the general circle equation:

$$(x-h)^2 + (y-k)^2 = r^2$$
$$(x-4)^2 + (y-6)^2 = 5^2$$

2-Find the circle's *x*– and *y*-intercepts.

To find the *x*-intercepts for any equation, you just plug in 0 for *y* and solve for *x*:

$$(x-4)^2 + (y-6)^2 = 5^2$$

$$(x-4)^2 + (0-6)^2 = 5^2$$
$$(x-4)^2 + 36 = 25$$
$$(x-4)^2 = -11$$

You can't square something and get a negative number, so this equation has no solution; therefore, the circle has no *x*-intercepts. (Of course, you can just look at the figure and see that the circle doesn't intersect the *x*-axis, but it's good to know how the math confirms this.

To find the *y*-intercepts, plug in 0 for *x* and solve for *y*:

$$(0-4)^2 + (y-6)^2 = 5^2$$
$$16 + (y-6)^2 = 25$$
$$(y-6)^2 = 9$$
$$y-6 = \pm\sqrt{9}$$
$$y = \pm 3 + 6$$
$$y = 3 \text{ or } 9$$

Thus, the circle's *y*-intercepts are (0, 3) and (0, 9).

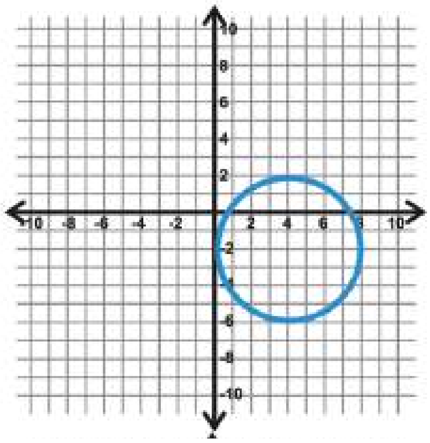
EXERCISES

Find the center and radius of each circle. Then, graph each circle.

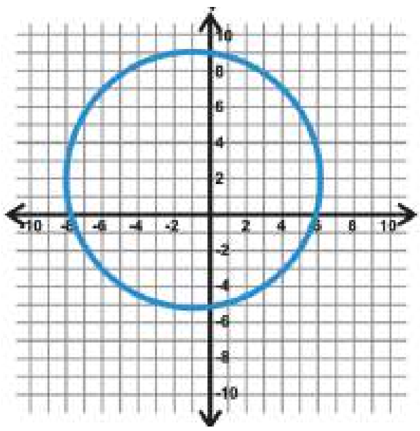
1. $(x+5)^2+(y-3)^2=16$
2. $x^2+(y+8)^2=4$
3. $(x-7)^2+(y-10)^2=20$
4. $(x+2)^2+y^2=8$

Find the equation of the circles below.

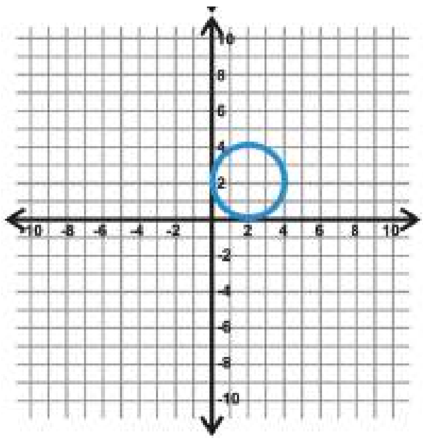
5.



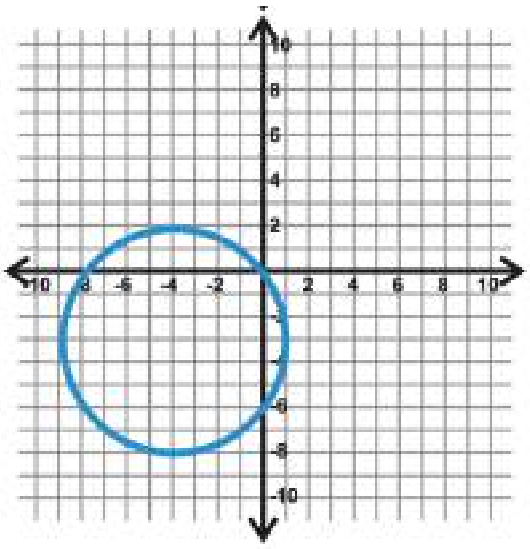
6.



7.



8.



9. Determine if the following points are on $(x+1)^2+(y-6)^2=45$.

   a. (2, 0)

   b. (-3, 4)

   c. (-7, 3)

Find the equation of the circle with the given center and point on the circle.

10. center: (2, 3), point: (-4, -1)

11. center: (10, 0), point: (5, 2)

12. center: (-3, 8), point: (7, -2)

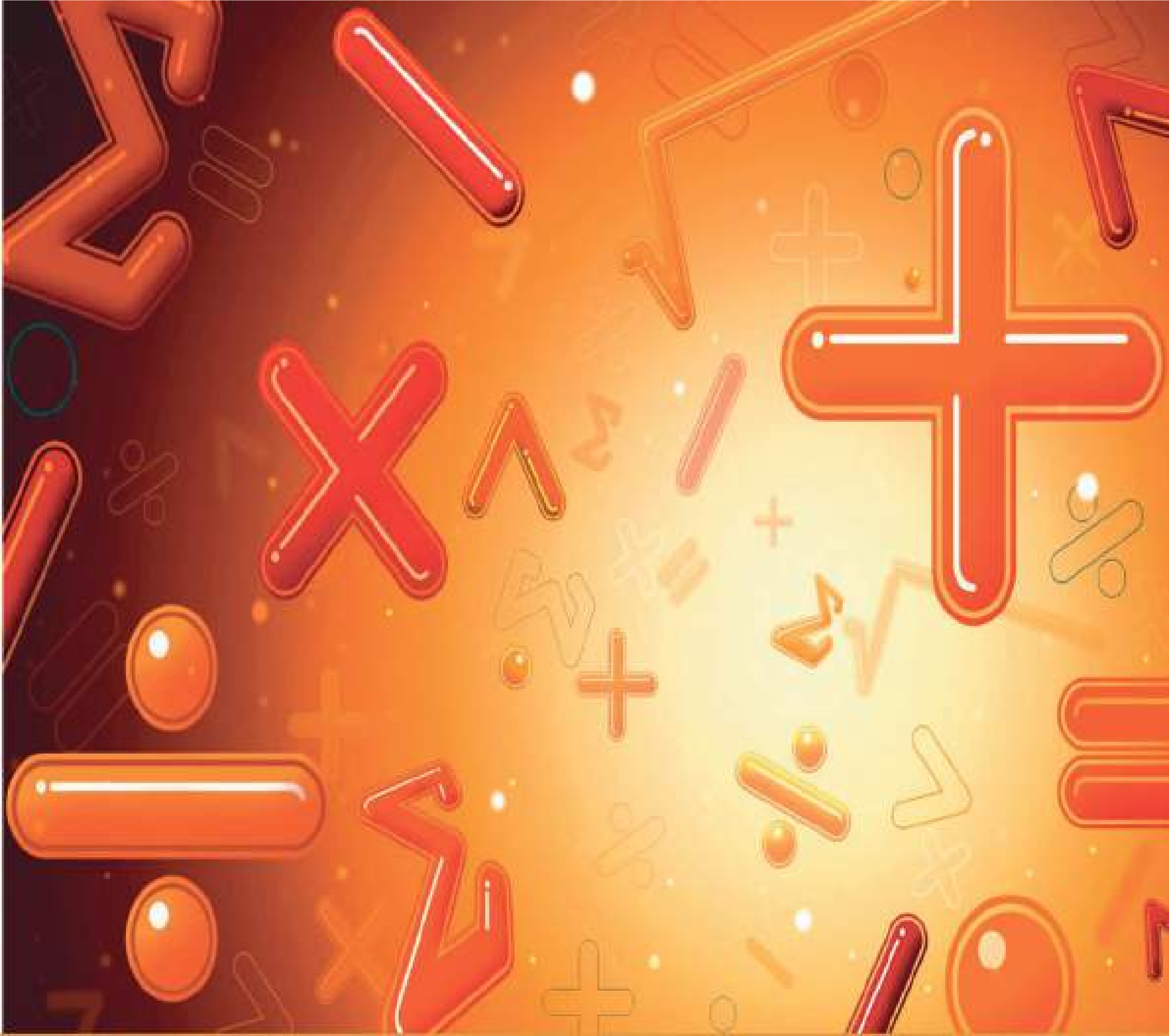
13. center: (6, -6), point: (-9, 4)

Find the equations of the circles

14. A(−12,−21),B(2,27) and C(19,10)

15. A(−2,5),B(5,6) and C(6,−1)

16. A(−11,−14),B(5,16) and C(12,9)



Chapter 4

CIRCULAR FRACTION

INTRODUCTION TO TRIGONOMETRY

A. ANGLES AND THE UNIT CIRCLE

In your previous studies you have already learned the basic principles of trigonometry on the unit circle and in right triangles. Before we begin our study of trigonometry, it will be helpful to review these basic concepts and definitions.

Definition

angle, directed angle

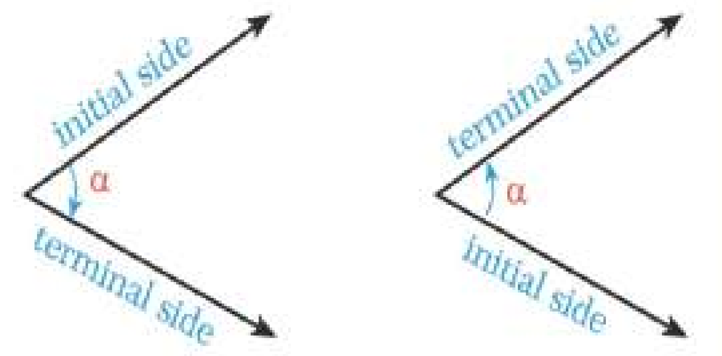
An **angle** is the union of two rays which have the same initial point.

If one of the rays of an angle is called the initial side of the angle and other ray is called the terminal side, then the angle is called a **directed angle**.

Definition

negative angle, positive angle

If a directed angle is measured in a clockwise direction from its initial side then the angle is a **negative angle**. If the angle is measured in a counterclockwise direction then it is a **positive angle**. In trigonometry we use both positive and negative angles.



negative angle: $\alpha = -60°$ positive angle: $\alpha = 60°$

We can measure angles using different units of measurement. The most common units are **degrees** and **radians**. We write $^\circ$ to show a degree measurement: one full circle measures 360°. We write R to to show a radian measurement: one full circle measures $2\pi^R$. We can also omit the radian notation if it is clear that an angle is in radians: $\alpha = \frac{3\pi}{2}$ means that the angle α measures $\frac{3\pi}{2}$ radians.

We can use a simple formula to convert between degree (D) and radian (R) measures:

$$\frac{D}{180^\circ} = \frac{R}{\pi}$$

For example, $360^\circ = 2\pi^R$, $90^\circ = \frac{\pi}{2}$, $45^\circ = \frac{\pi}{4}$, ...

In trigonometry we often work with angles drawn in the coordinate plane.

Definition

standard position

An angle in the coordinate plane is in **standard position** if its vertex is at the origin of the plane and its initial side lies along the positive x-axis.

Definition

coterminal angles

If two or more angles in standard position have coincident terminal sides then they are called **coterminal angles**. For example, 90° and -270° are coterminal angles. 180° and -180° are also coterminal angles.

Definition

primary directed angle

Let β be an angle which is greater than 360° or less than 0°. Then α is called the **primary directed angle** of β if α is coterminal with β and α ∈ [0°, 360°). In other words, α is the angle between 0° and 360° which is coterminal with β.

We can write: $\beta = \alpha \pm k \cdot 360°$, i.e. $\beta = \alpha \pm 2k\pi$.

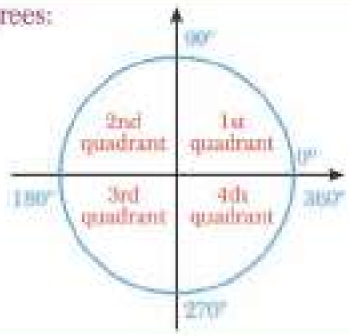
Definition

unit circle, quadrant

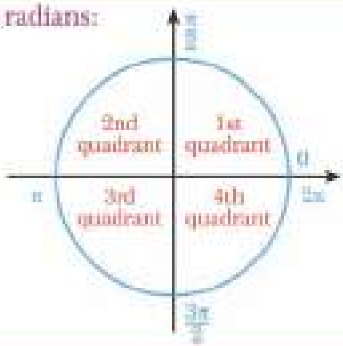
The circle whose center lies at the origin of the coordinate plane and whose radius is 1 unit is called the **unit circle**.

The coordinate axes divide the unit circle into four parts, called **quadrants**. The quadrants are numbered in a counterclockwise direction.

in degrees:



in radians:



Definition

quadrantal angles

The intersection points of the unit circle and the coordinate axes correspond to angles measured on the circle. These angles are called **quadrantal angles**. In other words, 0°, 90°, 180°, 270°, 360°, ... and 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, 2π, ... are quadrantal angles.

In the unit circle, if $\alpha \in \left(0, \frac{\pi}{2}\right)$ then α is in the first quadrant, if $\alpha \in \left(\frac{\pi}{2}, \pi\right)$ then it is in the second quadrant, if $\alpha \in \left(\pi, \frac{3\pi}{2}\right)$ then it is in the third quadrant and if $\alpha \in \left(\frac{3\pi}{2}, 2\pi\right)$ then it is in the fourth quadrant. The same applies to the equivalent intervals in degrees.

If an angle is greater than 360° = 2π or less than 0°, we can find its quadrant by first finding its primary directed angle.

EXAMPLE 1 In which quadrant does each angle lie?

a. 75° b. 228° c. 305° d. 740° e. –442° f. $\frac{7\pi}{3}$ g. $-\frac{17\pi}{5}$

Solution

a. 75° < 90°, so it is in the first quadrant.

b. 228° ∈ (180°, 270°), so it is in the third quadrant.

c. 305° ∈ (270°, 360°), so it is in the fourth quadrant.

d. 740° = 20° + (2 · 360°) ≡ 20° and 20° ∈ (0°, 90°). So 740° is in the first quadrant.

e. –442° = 278° – (2 · 360°) ≡ 278° and 278° ∈ (270°, 360°). So –442° is in the fourth quadrant.

f. $\frac{7\pi}{3} = \frac{\pi}{3} + 2\pi \equiv \frac{\pi}{3}$ and $\frac{\pi}{3} \in \left(0, \frac{\pi}{2}\right)$, so $\frac{7\pi}{3}$ is in the first quadrant.

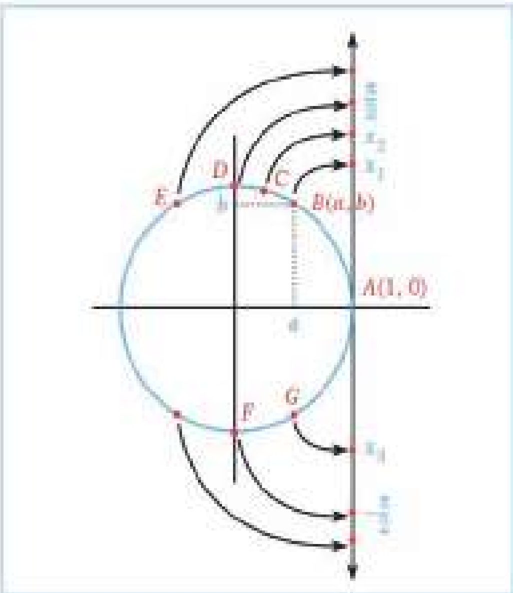
g. $-\frac{17\pi}{5} = \frac{3\pi}{5} - (2 \cdot 2\pi) \equiv \frac{3\pi}{5}$ and $\frac{3\pi}{5} \in \left(\frac{\pi}{2}, \pi\right)$, so $-\frac{17\pi}{5}$ is in the second quadrant.

We write ≡ to show that two angles are coterminal: α ≡ β means that α and β are coterminal.

Be careful! α ≡ β does not mean α = β. For example, 740° ≡ 20° but 740° ≠ 20°.

Remember that there is a correspondence between the points on the unit circle and the real number line.

Imagine the real number line placed vertically next to the unit circle, so that O on the real number line coincides with $A(1, 0)$ on the unit circle. Then we can 'wrap' the real number line around the top and bottom halves of the circle to see the correspondences: in the figure, we can see that point D corresponds to $\frac{\pi}{2} \cong 1.571$ on the number line, point F corresponds to $-\frac{\pi}{2}$, point $B(a, b)$ corresponds to x_1, and so on. The intercept of one radian and the unit circle corresponds to 1 on the real number line.



EXAMPLE 2 Find the real number which corresponds to the central angle 60° on the unit circle, using $\pi \cong 3.14$.

Solution From the formula $\frac{D}{180°} = \frac{R}{\pi}$ we have $\frac{60°}{180°} = \frac{R}{\pi}$, so $R = \frac{60° \cdot \pi}{180°} = \frac{\pi}{3}$. Using $\pi \cong 3.14$ gives us $R \cong \frac{3.14}{3} \cong 1.047$. This is the required real number.

We sometimes use Roman numerals to identify the quadrants:
quadrant I = first quadrant
quadrant II = second quadrant
quadrant III = third quadrant
quadrant IV = fourth quadrant.

Check Yourself

1. Determine the quadrant for each angle.

   a. 175° b. 328° c. 215° d. –395° e. –740 f. $\frac{127\pi}{7}$ g. $-\frac{25\pi}{6}$

2. Find the real number which corresponds to each central angle on the unit circle, correct to three decimal places.

   a. 25° b. 135° c. –80° d. $\frac{2\pi}{5}$ e. $\frac{16\pi}{3}$

Answers

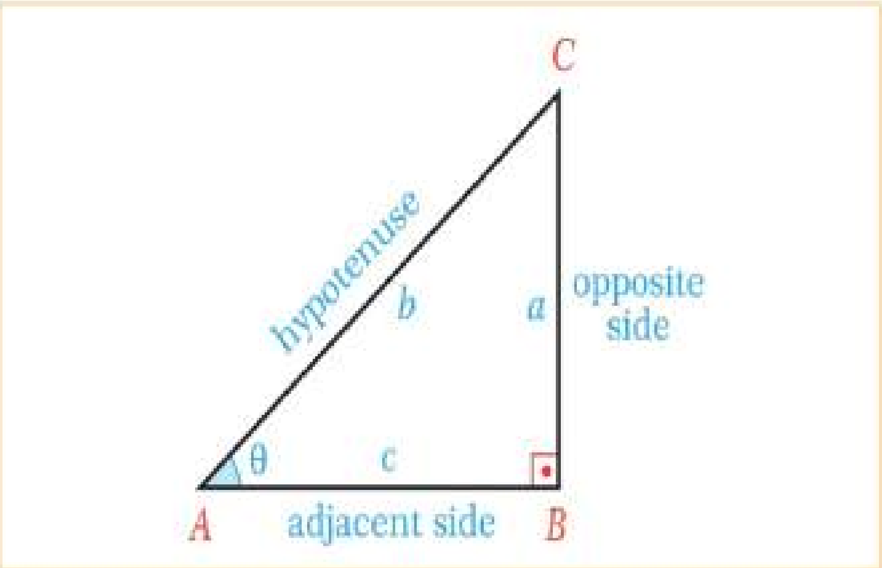
1. a. II b. IV c. III d. IV e. IV f. I g. IV

2. a. 0.436 b. 2.356 c. –1.396 d. 1.256 e. 16.75

B. BASIC TRIGONOMETRIC RATIOS

Look at the triangle opposite. By triangle similarity we know that if θ remains constant then the ratios of the side lengths of the triangle remain constant, even if the triangle gets bigger or smaller. These ratios are called **trigonometric ratios**, and each trigonometric ratio has a special name.

In a right triangle:



Some books write tg θ for tan θ, ctg θ for cot θ and cosec θ for csc θ.

the **sine** of angle $\theta = \sin\theta = \dfrac{\textit{opposite side}}{\textit{hypotenuse}} = \dfrac{a}{b}$,

the **cosine** of angle $\theta = \cos\theta = \dfrac{\textit{adjacent side}}{\textit{hypotenuse}} = \dfrac{c}{b}$,

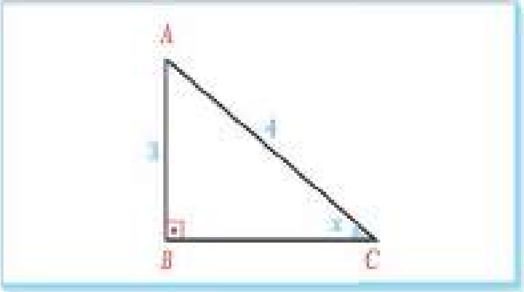
the **tangent** of angle $\theta = \tan\theta = \dfrac{\textit{opposite side}}{\textit{adjacent side}} = \dfrac{a}{c}$,

the **cotangent** of angle $\theta = \cot\theta = \dfrac{\textit{adjacent side}}{\textit{opposite side}} = \dfrac{c}{a}$,

the **secant** of angle $\theta = \sec\theta = \dfrac{\textit{hypotenuse}}{\textit{adjacent side}} = \dfrac{b}{c}$,

the **cosecant** of angle $\theta = \csc\theta = \dfrac{\textit{hypotenuse}}{\textit{opposite side}} = \dfrac{b}{a}$.

EXAMPLE 3 In the figure, ΔABC is a right triangle. Given that $AB = 3$, $AC = 4$ and $m(\angle ACB) = x$, find the six trigonometric ratios for x.



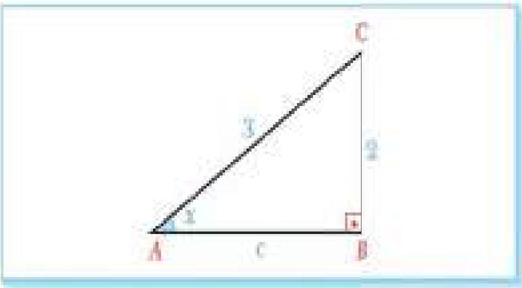
Solution We can find BC by using the Pythagorean Theorem:

$3^2 + BC^2 = 4^2$ and $BC^2 = 16 - 9 = 7$, so $BC = \sqrt{7}$. Now $a = 3$, $b = 4$, and $c = \sqrt{7}$, so

$$\sin x = \frac{3}{4},\ \cos x = \frac{\sqrt{7}}{4},\ \tan x = \frac{3}{\sqrt{7}} = \frac{3\sqrt{7}}{7},\ \cot x = \frac{\sqrt{7}}{3},\ \sec x = \frac{4}{\sqrt{7}} = \frac{4\sqrt{7}}{7} \text{ and } \csc x = \frac{4}{3}.$$

EXAMPLE 4 In a right triangle, $\sin x = \frac{2}{3}$. Find $\frac{\cos x + \tan x}{\cot x}$.

Solution Let us draw the right triangle.



$\sin x = \frac{\textit{opposite side}}{\textit{hypotenuse}} = \frac{2}{3}$, so we can say that the side opposite the angle has length 2 and the hypotenuse is 3. We can find the adjacent side by using the Pythagorean Theorem:

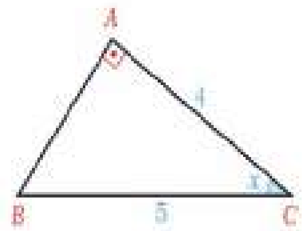
$c^2 + 2^2 = 3^2$ so $c = \sqrt{5}$.

So $\cos x = \frac{\sqrt{5}}{3}$, $\tan x = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$ and $\cot x = \frac{\sqrt{5}}{2}$, which gives us

$$\frac{\cos x + \tan x}{\cot x} = \frac{\frac{\sqrt{5}}{3} + \frac{2\sqrt{5}}{5}}{\frac{\sqrt{5}}{2}} = \frac{11\sqrt{5}}{15} \cdot \frac{2}{\sqrt{5}} = \frac{22}{15}.$$

Check Yourself

1. In a right triangle, $\tan x = \frac{1}{2}$. Find $\frac{\cos x \cdot \sin x}{\cot x}$.
2. In the figure at the right, ΔABC is a right triangle. Given that $AC = 4$, $BC = 5$ and $m(\angle ACB) = x$, find $\frac{\sin x \cdot \cos x}{\tan x + \cot x}$.



Answers

1. $\frac{1}{5}$ 2. $\frac{144}{625}$

C. TRIGONOMETRIC IDENTITIES

Remember!
An identity is an equation which is true for all possible values of its variable(s).

Remember!
$\sin^2 x$ means $(\sin x)\cdot(\sin x)$,
$\cos^2 x$ means $(\cos x)\cdot(\cos x)$,
etc.

The trigonometric ratios are related to each other by equations called **trigonometric identities**. The properties of a right triangle give us the following identities:

1. $\sin^2 x + \cos^2 x = 1$

   From this identity we also get $\sin^2 x = 1 - \cos^2 x$ and $\cos^2 x = 1 - \sin^2 x$.

2. $\tan^2 x + 1 = \sec^2 x$
3. $\cot^2 x + 1 = \csc^2 x$
4. $\tan x = \dfrac{\sin x}{\cos x}$
5. $\cot x = \dfrac{\cos x}{\sin x}$
6. $\tan x \cdot \cot x = 1$

   From this identity we also get $\tan x = \dfrac{1}{\cot x}$ and $\cot x = \dfrac{1}{\tan x}$.

7. $\sec x = \dfrac{1}{\cos x}$
8. $\csc x = \dfrac{1}{\sin x}$.

We can use these identities to simplify trigonometric expressions and verify equations.

Note

Simplifying an expression generally means changing ratios in $\tan x$, $\cot x$, $\sec x$ and $\csc x$ to ratios in $\sin x$ and $\cos x$. We can also factorize an expression to write it more simply.

EXAMPLE 5 Simplify $\dfrac{\sin x \cdot \cos x \cdot \tan x}{\csc x}$.

Solution Let us use the identities $\tan x = \dfrac{\sin x}{\cos x}$ and $\csc x = \dfrac{1}{\sin x}$.

Then the expression becomes $\dfrac{\sin x \cdot \cos x \cdot \dfrac{\sin x}{\cos x}}{\dfrac{1}{\sin x}} = \sin^3 x$.

So the simplified form is $\sin^3 x$.

EXAMPLE 6 Simplify $\dfrac{\tan x + \cot x}{\sec x \cdot \csc x}$.

Solution Let us use the identities $\tan x = \dfrac{\sin x}{\cos x}$, $\cot x = \dfrac{\cos x}{\sin x}$, $\sec x = \dfrac{1}{\cos x}$ and $\csc x = \dfrac{1}{\sin x}$. Then

$$\frac{\tan x + \cot x}{\sec x \cdot \csc x} = \frac{\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}}{\frac{1}{\cos x} \cdot \frac{1}{\sin x}} = \frac{\frac{\sin^2 x + \cos^2 x}{\sin x \cdot \cos x}}{\frac{1}{\sin x \cdot \cos x}} = \frac{1}{\sin x \cdot \cos x} \cdot \frac{\sin x \cdot \cos x}{1} = 1.$$

So the simplified form is 1.

EXAMPLE 7 Verify the identity $\tan^2 x - \sin^2 x = \tan^2 x \cdot \sin^2 x$.

Solution Let us work on the left-hand side of the equation.

$$\tan^2 x - \sin^2 x = \frac{\sin^2 x}{\cos^2 x} - \sin^2 x$$

$$= \sin^2 x \cdot \left(\frac{1}{\cos^2 x} - 1\right) \qquad \text{(take out } \sin^2 x\text{)}$$

$$= \sin^2 x \cdot \left(\frac{1 - \cos^2 x}{\cos^2 x}\right) \qquad \text{(equalize the denominators)}$$

$$= \frac{\sin^2 x \cdot \sin^2 x}{\cos^2 x} \qquad \text{(use } 1 - \cos^2 x = \sin^2 x\text{)}$$

$$= \frac{\sin^2 x}{\cos^2 x} \cdot \sin^2 x$$

$= \tan^2 x \cdot \sin^2 x$. This is the required result.

Check Yourself

1. Simplify each trigonometric expression.

   a. $\sin x \cdot \cot x \cdot \sec x$ b. $\dfrac{\sin x \cdot \cos x}{\tan x} + \dfrac{\sin x}{\csc x}$ c. $\dfrac{1 + \sin x}{1 + \csc x}$

   d. $\dfrac{1 + \sin x}{\cos x} - \dfrac{\cos x}{1 - \sin x}$ e. $\dfrac{(\sin x + \cos x)^2 - 1}{(\sin x - \cos x)^2 - 1}$

2. Verify the identities.

   a. $\dfrac{\cos x}{1 - \sin x} + \dfrac{1 - \sin x}{\cos x} = 2\sec x$ b. $\dfrac{\sin x}{1 - \cos x} - \cot x = \csc x$

Answers

1. a. 1 b. 1 c. $\sin x$ d. 0 e. –1 2. Hint: Work on the left-hand side of the equations.

D. TRIGONOMETRIC THEOREMS

1. Basic Theorems

Theorem

law of cosines

Remember!

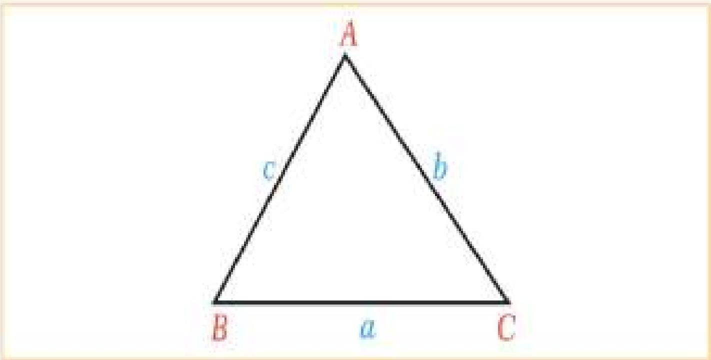
a is the side opposite $\angle A$, b is the side opposite $\angle B$, etc.

In any triangle ABC,

$$a^2 = b^2 + c^2 - 2bc \cdot \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cdot \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cdot \cos C.$$



EXAMPLE 8 A triangle has side lengths $a = 5$ cm and $b = 4$ cm and angle $m(\angle C) = 60°$. Find the length of side c.

Solution By the law of cosines, $c^2 = a^2 + b^2 - 2ab \cdot \cos C$

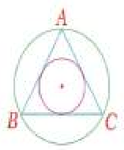
$$= 5^2 + 4^2 - (2 \cdot 5 \cdot 4 \cdot \cos 60°)$$

$$= 25 + 16 - (2 \cdot 20 \cdot \frac{1}{2}) = 21. \text{ So } c = \sqrt{21}.$$

Theorem

law of sines

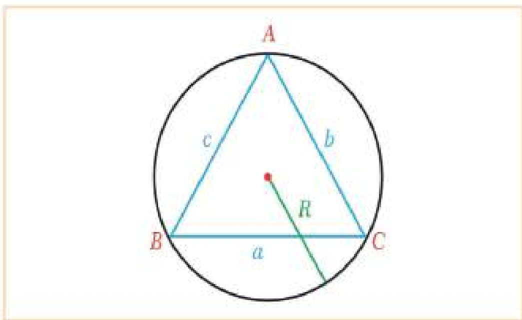
Inscribed and circumscribed:



The green circle is the circumscribed circle of ΔABC. The pink circle is the inscribed circle of ΔABC.

If R is the radius of the circumscribed circle of a triangle ABC with side lengths a, b and c, then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$



EXAMPLE 9 In a triangle ABC, $a = 5$ cm, $m(\angle A) = 30°$ and $m(\angle C) = 105°$. Find the length of side b.

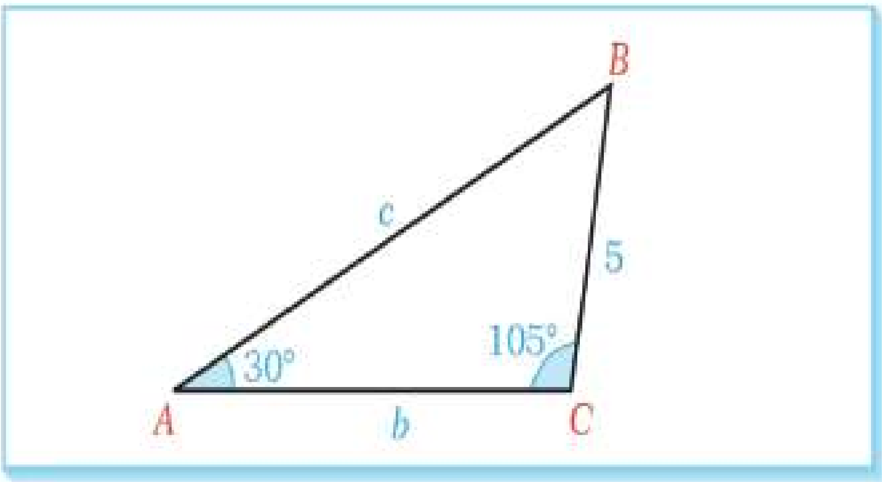
Solution First let us find $m(\angle B)$:

$$m(\angle B) = 180° - m(\angle A) - m(\angle C)$$
$$= 180° - 30° - 105° = 45°.$$

Now by the law of sines we have

$$\frac{a}{\sin A} = \frac{b}{\sin B}, \text{ i.e. } \frac{5}{\sin 30°} = \frac{b}{\sin 45°}$$

$$\Rightarrow \frac{5}{\frac{1}{2}} = \frac{b}{\frac{\sqrt{2}}{2}}, \text{ so } b = 5\sqrt{2} \text{ cm}.$$



Theorem

law of tangents

In any triangle ABC with side lengths a, b, c and interior angles A, B, C,

$$\frac{a+b}{a-b}=\frac{\tan(\frac{A+B}{2})}{\tan(\frac{A-B}{2})},\ \frac{a+c}{a-c}=\frac{\tan(\frac{A+C}{2})}{\tan(\frac{A-C}{2})}\text{ and }\frac{b+c}{b-c}=\frac{\tan(\frac{B+C}{2})}{\tan(\frac{B-C}{2})}.$$

E. TRIGONOMETRIC FORMULAS

Sometimes we can find the value of a trigonometric ratio by writing it as the sum or difference of more familiar trigonometric ratios. The properties of a 30°-60°-90° triangle and a 45°-45°-90° triangle give us the common ratios in the table at the right. We can use these ratios with the following formulas.

x	$\sin x$	$\cos x$	$\tan x$
$30° = \frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$45° = \frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$60° = \frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

1. Sum and Difference Formulas

Property

sum and difference formulas

$\sin(x+y) = (\sin x \cdot \cos y) + (\cos x \cdot \sin y)$

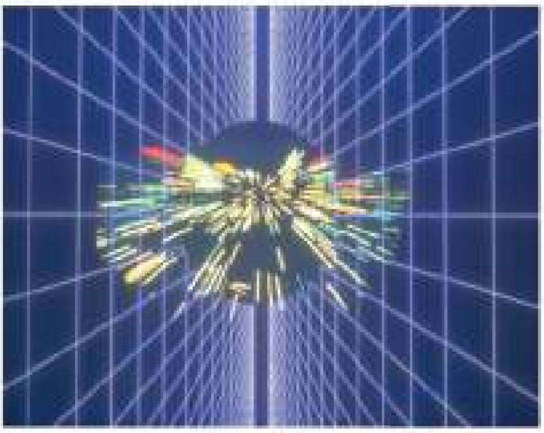
$\sin(x-y) = (\sin x \cdot \cos y) - (\cos x \cdot \sin y)$

$\cos(x+y) = (\cos x \cdot \cos y) - (\sin x \cdot \sin y)$

$\cos(x-y) = (\cos x \cdot \cos y) + (\sin x \cdot \sin y)$

$$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\cdot\tan y}$$

$$\tan(x-y)=\frac{\tan x-\tan y}{1+\tan x\cdot\tan y}$$



Note

We can use the identity $\cot x = \dfrac{1}{\tan x}$ to solve problems involving $\cot x$. Therefore we do not need to remember the sum and difference formulas for $\cot x$ explicitly.

EXAMPLE 10 $\cos 75° = ?$

Solution $\cos 75° = \cos(45° + 30°) = (\cos 45° \cdot \cos 30°) - (\sin 45° \cdot \sin 30°)$

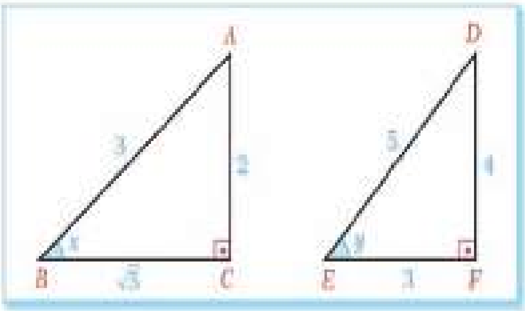
$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}.$$

EXAMPLE 11 $\tan 105° = ?$

Solution $$\tan 105° = \tan(60° + 45°) = \frac{\tan 45° + \tan 60°}{1 - \tan 45° \cdot \tan 60°} = \frac{1+\sqrt{3}}{1-1\cdot\sqrt{3}} = \frac{1+\sqrt{3}}{1-\sqrt{3}}.$$

EXAMPLE 12 Given that $\sin x = \dfrac{2}{3}$ and $\cos y = \dfrac{3}{5}$, find $\sin(x-y)$.

Solution We can draw right triangles to find the other trigonometric ratios of the angles x and y. By the Pythagorean Theorem, we get $BC = \sqrt{5}$ and $DF = 4$. So



$\sin(x-y) = \sin x \cdot \cos y - \cos x \cdot \sin y$

$$= \frac{2}{3}\cdot\frac{3}{5} - \frac{\sqrt{5}}{3}\cdot\frac{4}{5} = \frac{6-4\sqrt{5}}{15}.$$

2. Double-Angle and Half-Angle Formulas

Property

double-angle formulas

Substituting $y = x$ in the sum and difference formulas gives us

$\sin 2x = 2 \cdot \sin x \cdot \cos x$

$\cos 2x = \cos^2 x - \sin^2 x$

$= 2\cos^2 x - 1$

$= 1 - 2\sin^2 x$

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}.$$

These are called the **double-angle formulas**.

If we replace x with $\frac{x}{2}$, we get the **half-angle formulas**:

$$\sin x = 2 \cdot \sin\frac{x}{2} \cdot \cos\frac{x}{2}$$

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = 2\cos^2\frac{x}{2} - 1 = 1 - 2\sin^2\frac{x}{2}$$

$$\tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}$$

EXAMPLE 13 $\sin x = \frac{1}{3}$ is given. Find $\cos 2x$.

Solution $\cos 2x = 1 - 2\sin^2 x = 1 - 2\cdot(\frac{1}{3})^2 = 1 - \frac{2}{9} = \frac{7}{9}$.

EXAMPLE 14 Given that $\sin x + \cos x = \frac{2}{3}$, find $\sin 2x$.

Solution Squaring both sides of $\sin x + \cos x = \frac{2}{3}$ gives us

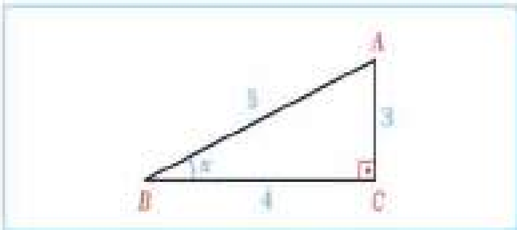
$$\sin^2 x + (2\cdot\sin x\cdot\cos x) + \cos^2 x = \frac{4}{9}.$$

Use $\sin^2 x + \cos^2 x = 1$ and $2\sin x \cdot \cos x = \sin 2x$. Then

$$1 + \sin 2x = \frac{4}{9}, \text{ and so } \sin 2x = \frac{4}{9} - 1 = -\frac{5}{9}.$$

EXAMPLE 15 Find $\cos\frac{x}{2}$ if $\sin x = 0.6$.

Solution First we draw a right triangle to find $\cos x$, using $\sin x = 0.6 = \frac{3}{5}$.



By the Pythagorean Theorem, $BC = 4$ and so $\cos x = \frac{4}{5}$.

Now by the half-angle formula $\cos x = 2\cos^2\frac{x}{2} - 1$, we have

$$\cos\frac{x}{2} = \sqrt{\frac{\cos x + 1}{2}} = \sqrt{\frac{\frac{4}{5}+1}{2}} = \sqrt{\frac{9}{10}} = \frac{3\sqrt{10}}{10}.$$

Check Yourself

1. Calculate each ratio.

   a. $\sin 15°$ b. $\tan 15°$ c. $\cos 105°$ d. $\cot 75°$

2. Given that $\tan x = \frac{4}{3}$ and $\cot y = \frac{5}{12}$, find $\sin(x + y)$, $\cos(x - y)$ and $\cot(x - y)$.

3. $\cos 2x = \frac{1}{4}$ is given. Find $\sin x$ and $\tan x$.

4. Given $\sin x - \cos x = \frac{1}{2}$, find $\cos 2x$.

5. $\sin 12° = a$ is given. Find $\cos 24°$ in terms of a.

Answers

1. a. $\frac{\sqrt{6}-\sqrt{2}}{4}$ b. $2-\sqrt{3}$ c. $\frac{\sqrt{2}-\sqrt{6}}{4}$ d. $2-\sqrt{3}$

2. $\sin(x+y) = \frac{56}{65}$, $\cos(x-y) = \frac{63}{65}$, $\cot(x-y) = -\frac{63}{16}$ 3. $\sin x = \frac{\sqrt{6}}{4}$, $\tan x = \frac{\sqrt{15}}{5}$

4. $\frac{\sqrt{7}}{4}$ 5. $1-2a^2$

3. Reduction Formulas

The trigonometric reduction formulas help us to 'reduce' a trigonometric ratio to a ratio of an acute angle. If the acute angle is a common angle, this technique helps us to find the ratio. For example, imagine you need to find $\cot 300°$.

We can say that $300° = 270° + 30°$.

By the reduction formula for the cotangent, $\cot 300° = -\tan 30° = -\frac{\sqrt{3}}{3}$.

To derive the reduction formulas, first we need to know the signs of the trigonometric functions in each quadrant:

1.

		$\sin x$	$\cos x$	$\tan x$	$\cot x$
first quadrant	$\left(0, \frac{\pi}{2}\right)$	+	+	+	+
second quadrant	$\left(\frac{\pi}{2}, \pi\right)$	+	–	–	–
third quadrant	$\left(\pi, \frac{3\pi}{2}\right)$	–	–	+	+
fourth quadrant	$\left(\frac{3\pi}{2}, 2\pi\right)$	–	+	–	–

2. If we have $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ in the reduction formula, the formula changes sine to cosine and tangent to cotangent. If we have π or 2π in the formula, the function does not change.

3. Now we can combine these two pieces of information to get the reduction formulas:

$\sin(\frac{\pi}{2}-x)=\cos x,\ \cos(\frac{\pi}{2}-x)=\sin x,\ \tan(\frac{\pi}{2}-x)=\cot x$

$\sin(\frac{\pi}{2}+x)=\cos x,\ \cos(\frac{\pi}{2}+x)=-\sin x,\ \tan(\frac{\pi}{2}+x)=-\cot x$

$\sin(\frac{3\pi}{2}-x)=-\cos x,\ \cos(\frac{3\pi}{2}-x)=-\sin x,\ \tan(\frac{3\pi}{2}-x)=\cot x$

$\sin(\frac{3\pi}{2}+x)=-\cos x,\ \cos(\frac{3\pi}{2}+x)=\sin x,\ \tan(\frac{3\pi}{2}+x)=-\cot x$

$\sin(\pi-x)=\sin x,\ \cos(\pi-x)=-\cos x,\ \tan(\pi-x)=-\tan x$

$\sin(\pi+x)=-\sin x,\ \cos(\pi+x)=-\cos x,\ \tan(\pi+x)=\tan x$

$\sin(-x)=-\sin x,\ \cos(-x)=\cos x,\ \tan(-x)=-\tan x$

EXAMPLE 16 Simplify each expression, given that $0 < x < \frac{\pi}{2}$.

a. $\sin(\frac{\pi}{2}+x)$ b. $\cos(\frac{\pi}{2}+x)$ c. $\tan(\frac{3\pi}{2}+x)$ d. $\sin(2\pi-x)$ e. $\sin(\pi+x)$ f. $\cos(2\pi+x)$

Solution
a. $(\frac{\pi}{2}+x)$ is in the second quadrant, so $\sin(\frac{\pi}{2}+x)=\cos x$.
b. $(\frac{\pi}{2}+x)$ is in the second quadrant, so $\cos(\frac{\pi}{2}+x)=-\sin x$.
c. $(\frac{3\pi}{2}+x)$ is in the fourth quadrant, so $\tan(\frac{3\pi}{2}+x)=-\cot x$.
d. $(2\pi-x)$ is in the fourth quadrant, so $\cot(2\pi-x) = -\cot x$.
e. $(\pi+x)$ is in the third quadrant, so $\sin(\pi+x) = -\sin x$.
f. $(2\pi+x)$ is in the first quadrant, so $\cos(2\pi+x) = \cos x$.

EXAMPLE 17 Simplify $\frac{\cos(90°+x)+\sin(270°-x)+\sin(180°-x)}{\cos(-x)-\cos(360°-x)+\sin(90°+x)}$.

Solution Let us simplify each term using the reduction formulas:

$\cos(90°+x) = -\sin x$, $\sin(270°-x) = -\cos x$,

$\sin(180°-x) = \sin x$, $\cos(-x) = \cos x$,

$\cos(360°-x) = \cos x$ and $\sin(90°+x) = \cos x$. So

$$\frac{\cos(90°+x)+\sin(270°-x)+\sin(180°-x)}{\cos(-x)-\cos(360°-x)+\sin(90°+x)}=\frac{-\sin x-\cos x+\sin x}{\cos x-\cos x+\cos x}=\frac{-\cos x}{\cos x}=-1.$$

Periodicity and Graphs of the Trigonometric Functions

When an angle of measure θ and an angle of measure $\theta + 2\pi$ are in standard position, their terminal rays coincide. The two angles therefore have the same trigonometric function values:

$$\cos(\theta + 2\pi) = \cos\theta \qquad \sin(\theta + 2\pi) = \sin\theta \qquad \tan(\theta + 2\pi) = \tan\theta$$
$$\sec(\theta + 2\pi) = \sec\theta \qquad \csc(\theta + 2\pi) = \csc\theta \qquad \cot(\theta + 2\pi) = \cot\theta$$

Similarly, $\cos(\theta - 2\pi) = \cos\theta$, $\sin(\theta - 2\pi) = \sin\theta$, and so on. We describe this repeating behavior by saying that the six basic trigonometric functions are *periodic*.

DEFINITION Periodic Function

A function $f(x)$ is **periodic** if there is a positive number p such that $f(x + p) = f(x)$ for every value of x. The smallest such value of p is the **period** of f.

When we graph trigonometric functions in the coordinate plane, we usually denote the independent variable by x instead of θ. See Figure 1.73.

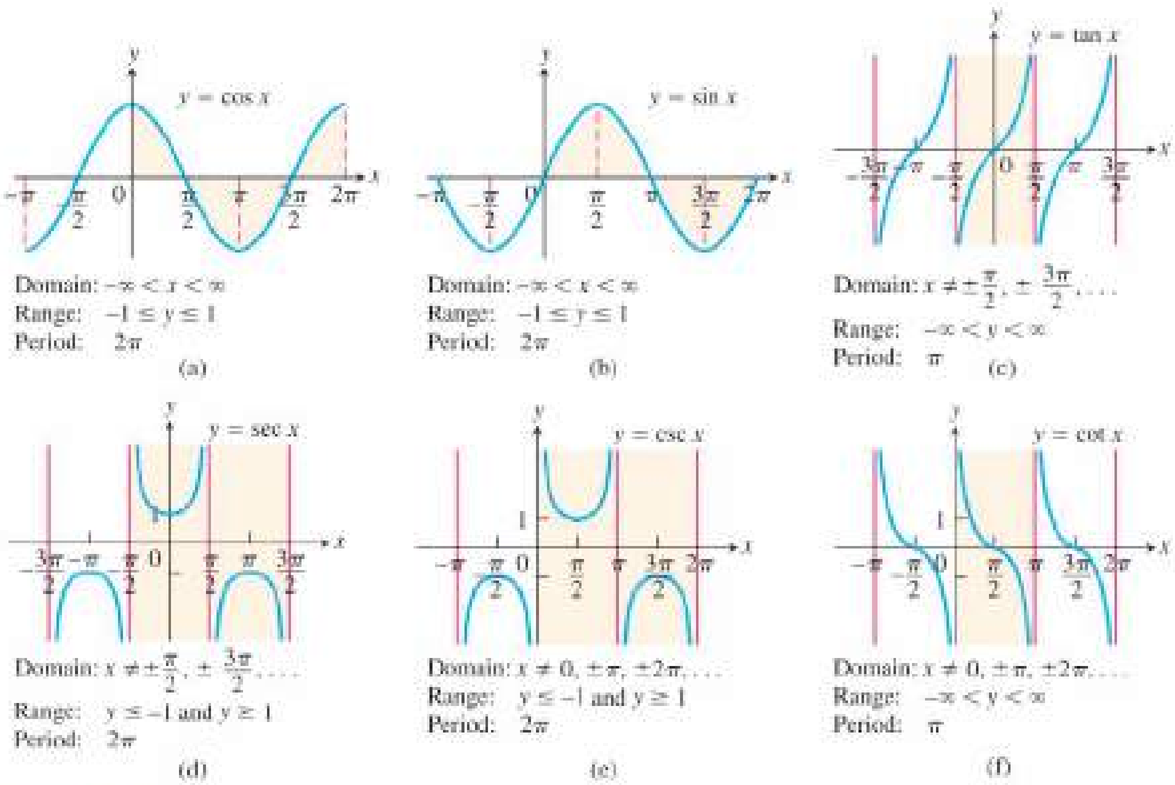


FIGURE 1.73 Graphs of the (a) cosine, (b) sine, (c) tangent, (d) secant, (e) cosecant, and (f) cotangent functions using radian measure. The shading for each trigonometric function indicates its periodicity.

Periods of Trigonometric Functions

Period π: $\tan(x + \pi) = \tan x$
$\cot(x + \pi) = \cot x$

Period 2π: $\sin(x + 2\pi) = \sin x$
$\cos(x + 2\pi) = \cos x$
$\sec(x + 2\pi) = \sec x$
$\csc(x + 2\pi) = \csc x$

As we can see in Figure 1.73, the tangent and cotangent functions have period $p = \pi$. The other four functions have period 2π. Periodic functions are important because many behaviors studied in science are approximately periodic. A theorem from advanced calculus says that every periodic function we want to use in mathematical modeling can be written as an algebraic combination of sines and cosines. We show how to do this in Section 11.11.

The symmetries in the graphs in Figure 1.73 reveal that the cosine and secant functions are even and the other four functions are odd:

Even	Odd
$\cos(-x) = \cos x$	$\sin(-x) = -\sin x$
$\sec(-x) = \sec x$	$\tan(-x) = -\tan x$
	$\csc(-x) = -\csc x$
	$\cot(-x) = -\cot x$

Identities

The coordinates of any point $P(x, y)$ in the plane can be expressed in terms of the point's distance from the origin and the angle that ray OP makes with the positive x-axis (Figure 1.69). Since $x/r = \cos\theta$ and $y/r = \sin\theta$, we have

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

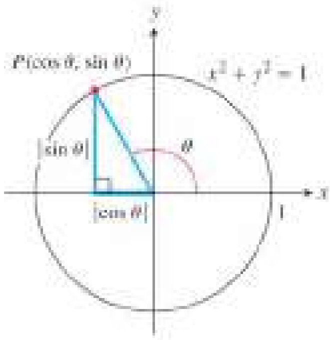


FIGURE 1.74 The reference triangle for a general angle θ.

When $r = 1$ we can apply the Pythagorean theorem to the reference right triangle in Figure 1.74 and obtain the equation

$$\cos^2\theta + \sin^2\theta = 1. \tag{1}$$

This equation, true for all values of θ, is the most frequently used identity in trigonometry. Dividing this identity in turn by $\cos^2\theta$ and $\sin^2\theta$ gives

$$1 + \tan^2\theta = \sec^2\theta.$$
$$1 + \cot^2\theta = \csc^2\theta.$$

The following formulas hold for all angles A and B (Exercises 53 and 54).

Addition Formulas

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$
$$\sin(A + B) = \sin A\cos B + \cos A\sin B \tag{2}$$

Substeraction Formulas

$$\cos(A - B) = \cos A\cos B + \sin A\sin B$$
$$\sin(A - B) = \sin A\cos B - \cos A\sin B \tag{3}$$

There are similar formulas for $\cos(A - B)$ and $\sin(A - B)$ (Exercises 35 and 36). All the trigonometric identities needed in this book derive from Equations (1) and (2). For example, substituting θ for both A and B in the addition formulas gives

Double-Angle Formulas

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$
$$\sin 2\theta = 2 \sin \theta \cos \theta \qquad (3)$$

Additional formulas come from combining the equations

$$\cos^2 \theta + \sin^2 \theta = 1, \qquad \cos^2 \theta - \sin^2 \theta = \cos 2\theta.$$

We add the two equations to get $2 \cos^2 \theta = 1 + \cos 2\theta$ and subtract the second from the first to get $2 \sin^2 \theta = 1 - \cos 2\theta$. This results in the following identities, which are useful in integral calculus.

Half-Angle Formulas

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2} \qquad (4)$$

$$\sin^2 \theta = \frac{1 - \cos 2\theta}{2} \qquad (5)$$

The Law of Cosines

If a, b, and c are sides of a triangle ABC and if θ is the angle opposite c, then

$$c^2 = a^2 + b^2 - 2ab \cos \theta. \qquad (6)$$

This equation is called the **law of cosines**.

We can see why the law holds if we introduce coordinate axes with the origin at C and the positive x-axis along one side of the triangle, as in Figure 1.75. The coordinates of A are $(b, 0)$; the coordinates of B are $(a \cos \theta, a \sin \theta)$. The square of the distance between A and B is therefore

$$\begin{aligned} c^2 &= (a \cos \theta - b)^2 + (a \sin \theta)^2 \\ &= a^2(\underbrace{\cos^2 \theta + \sin^2 \theta}_{1}) + b^2 - 2ab \cos \theta \\ &= a^2 + b^2 - 2ab \cos \theta. \end{aligned}$$

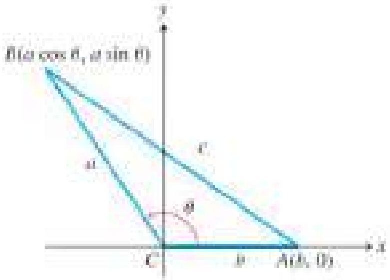


FIGURE 1.75 The square of the distance between A and B gives the law of cosines.

The law of cosines generalizes the Pythagorean theorem. If $\theta = \pi/2$, then $\cos \theta = 0$ and $c^2 = a^2 + b^2$.

EXERCISES

A. Angles and the Unit Circle

1. In which quadrant does each angle lie?

a. 275° b. 228° c. 185°

d. –530° e. $\frac{28\pi}{3}$ f. $\frac{125\pi}{6}$

2. Find the value of the real number which corresponds to each angle on the unit circle. Give your answer to three decimal places.

a. 45° b. 105° c. –70°

d. $\frac{3\pi}{7}$ e. $\frac{26\pi}{5}$

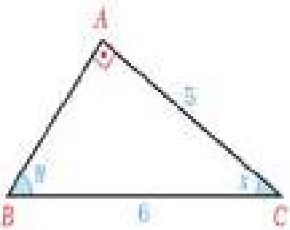
B. Basic Trigonometric Ratios

3. In a right triangle, $\cot x = \frac{1}{2}$. Find $\frac{\cos x \cdot \sin x}{\tan x}$.

4. In a right triangle, $\sin x = 0.4$. Find $\sin x + \cos x$.

5. In a right triangle, $\sec x = 3$. Find $\frac{\sin x + \cos x}{\csc x \cdot \tan x}$.

6. In the figure, ΔABC is a right triangle. Given that $AC = 5$, $BC = 6$, $m(\angle ACB) = x$ and $m(\angle ABC) = y$, find $\frac{(\sin x \cdot \cos x) \cdot (\tan y + \cot y)}{\sec y + \csc x}$.



C. Trigonometric Identities

7. Simplify each trigonometric expression.

a. $\cos x \cdot \tan x \cdot \csc x$ b. $\frac{\sin x \cdot \cos x}{\cot x} + \frac{\cos x}{\sec x}$

c. $\frac{1 - \cos x}{1 - \sec x}$ d. $\frac{1 + \cos x}{\sin x} - \frac{\sin x}{1 - \cos x}$

e. $\frac{(\tan x + \cot x)^2 - 2}{(\tan x - \cot x)^2 + 2}$ f. $\frac{\sec^2 x - \tan^2 x}{\csc^2 x - \cot^2 x}$

8. Verify each identity.

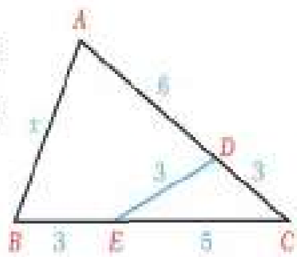
a. $\frac{1 + \tan^2 x}{\sec x \cdot \csc x} = \tan x$

b. $\frac{\sin^2 x - 1}{\cos^2 x - 1} + (\tan x \cdot \cot x) = \frac{1}{\sin^2 x}$

c. $\sin^2 x + \tan^2 x + \cos^2 x = \sec^2 x$

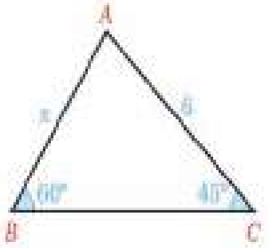
D. Trigonometric Theorems

9. In the figure, $AD = 6$, $DC = DE = BE = 3$ and $EC = 5$. Find the length $AB = x$.



10. A triangle has side lengths 4 cm, 5 cm and 7 cm. Find the radii r and R of the inscribed and circumscribed circles of this triangle.

11. In the figure, $AC = 6$, $m(\angle B) = 60°$ and $m(\angle C) = 45°$. Find the length $AB = x$ and the radius R of the triangle's circumscribed circle.



12. The sides of a triangle ABC are a, b and c. Write the ratio $\dfrac{\sin A - \sin B}{\sin A + \sin B}$ in terms of a and b.

13. The sides of a triangle ABC are a, b and c. Find $m(\angle C)$ in degrees if $c^2 = a^2 + b^2 + a2ab$.

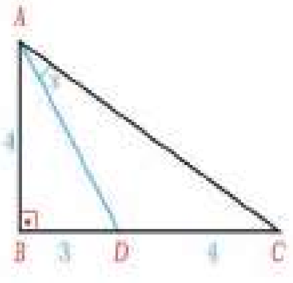
14. Calculate each ratio.

a. $\tan 75°$ b. $\sin 105°$

c. $\cos 15°$ d. $\cot 105°$

15. Given that $\tan x = \dfrac{1}{2}$ and $\cot y = \dfrac{1}{3}$, find $\sin(x + y)$ and $\cot(x - y)$.

16. In the figure, ABC is a right triangle. Given that $AB = 4$, $BD = 3$ and $DC = 4$, find $\tan x$.



17. $\tan x = 2$ is given. Find $\sin 2x + \cos 2x$.

18. $\sin 2x = \dfrac{3}{5}$ is given. Find $\sin x$, $\cos x$ and $\tan x$.

19. Find $\sin 2x$ if $\sin x + \cos x = \dfrac{\sqrt{2}}{3}$.

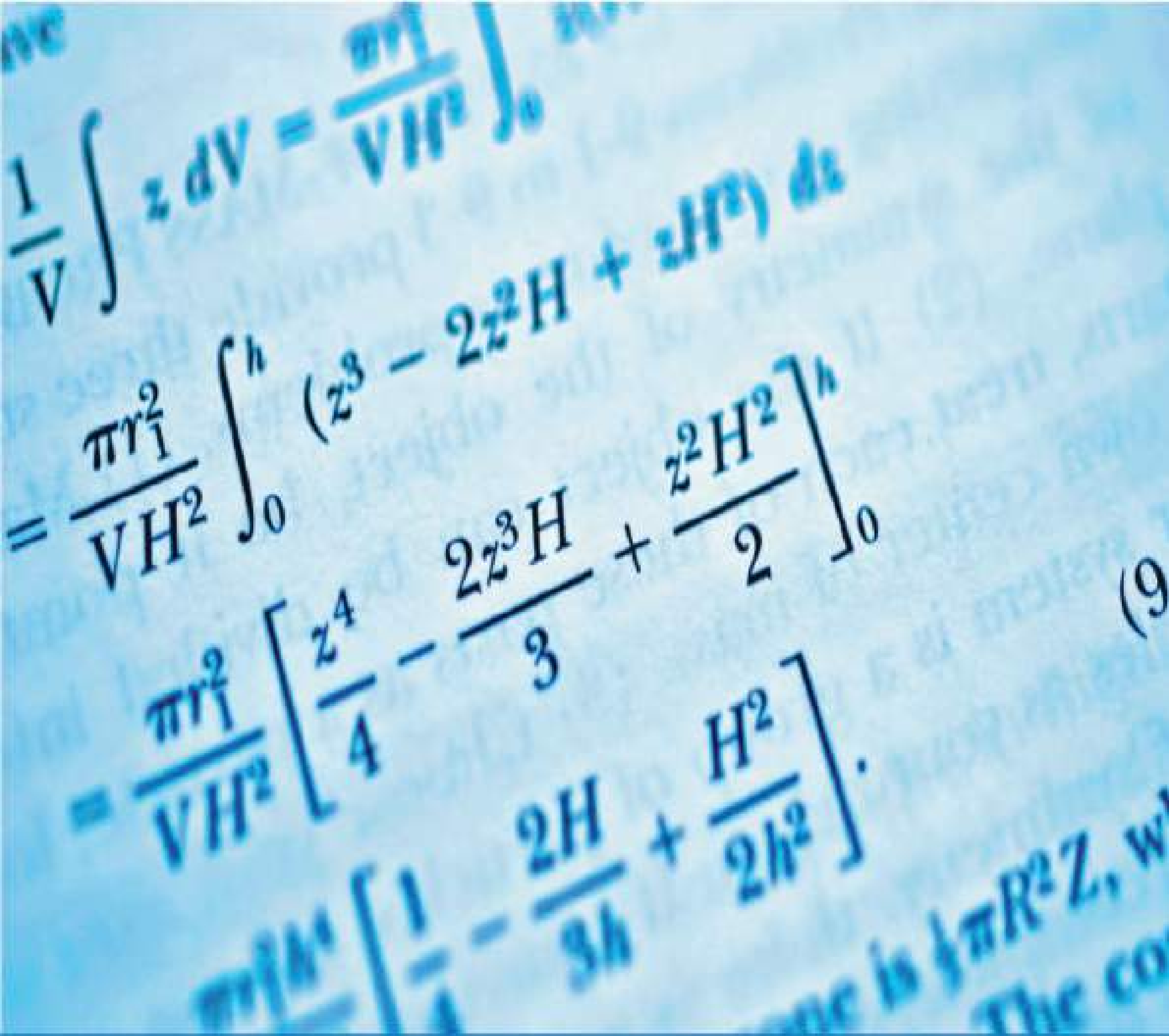
20. $\sin 18° = a$ is given. Find $\cos 54°$ in terms of a.

21. Find $\sin 2x$ if $\sin\dfrac{x}{2} = \dfrac{1}{3}$.

22. Calculate each ratio by using reduction formulas.

a. $\sin 210°$ b. $\cos 150°$ c. $\tan 225°$

d. $\cot 300°$ e. $\sin 390°$



Chapter 5

LIMIT

LIMIT OF A FUNCTION

A. LIMIT OF A POLYNOMIAL FUNCTION

Consider the polynomial function $f(x) = 2x$. We are asked to investigate what happens to the value of $f(x)$ as x gets closer to 2. We could begin by choosing a value of x which is close to 2, for example 1.5. We can calculate $f(1.5) = 2 \cdot 1.5 = 3$. Now we choose a value which is closer to 2, for example 1.75: $f(1.75) = 3.5$. Continuing like this, we can make a table of values of $f(x)$ as x gets closer to 2.

x	1.5	1.75	1.8	1.9	1.95	2	2.05	2.1	2.2	2.25	2.5
$2x$	3	3.5	3.6	3.8	3.9		4.1	4.2	4.4	4.5	5

Using this table, we can guess that as x gets closer to (i.e. *approaches*) 2, the value of $f(x)$ approaches 4. We say that 4 is the **limit** of $f(x) = 2x$ as x approaches 2, and write $\lim\limits_{x\to 2}(2x) = 4$. In this notation, the arrow symbol (→) means 'approaches'. $x \to 2$ means x approaches the number 2.

Notice that for $f(x) = 2x$, $\lim\limits_{x\to 2} f(x) = \lim\limits_{x\to 2} 2x = 4$, which is the same as $f(2)$. Similarly, we can calculate $\lim\limits_{x\to 3} f(x) = \lim\limits_{x\to 3} 2x = 6 = f(3)$ and $\lim\limits_{x\to 12} f(x) = \lim\limits_{x\to 12} 2x = 24 = f(12)$, etc. In other words, in each case $\lim\limits_{x\to c} f(x) = f(c)$. In fact, this result is true for any polynomial function.

Definition

limit of a polynomial function

The limit of a polynomial function $f(x)$ as x approaches a point c is $f(c)$: $\lim\limits_{x\to c} f(x) = f(c)$.

In other words, for $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$, $\lim\limits_{x\to c} f(x) = f(c) = a_n c^n + a_{n-1}c^{n-1} + \ldots + a_0$.

For example, let us calculate the limit of $f(x) = 2x$ when x approaches 5:

$\lim\limits_{x\to 5} f(x) = \lim\limits_{x\to 5} 2x = 2 \cdot 5 = 10$.

EXAMPLE 1 Calculate the limits.

a. $\lim\limits_{x\to 2}(4x-1)$ b. $\lim\limits_{x\to -1}(x^2+3x+2)$ c. $\lim\limits_{t\to 4}(3t-t^2)$

Solution These are all polynomial functions, so we can use $\lim\limits_{x\to c} f(x) = f(c)$.

a. $\lim\limits_{x\to 2}(4x-1) = 4\cdot 2 - 1 = 7$

b. $\lim\limits_{x\to -1}(x^2+3x+2) = (-1)^2 + 3\cdot(-1) + 2 = 1 - 3 + 2 = 0$

c. $\lim\limits_{t\to 4}(3t-t^2) = 3\cdot 4 - 4^2 = 12 - 16 = -4$.

Check Yourself

1. Given $f(x) = 4x - 1$, complete the table to find the limit of $f(x)$ as x approaches 3.

x	2.5	2.75	2.8	2.9	2.95	3	3.05	3.1	3.2	3.25	3.5
$f(x)$	9									12	

2. Calculate the limit of each polynomial function.

a. $\lim\limits_{x \to 0} 5x$ b. $\lim\limits_{x \to 3} x(2 - x)$ c. $\lim\limits_{x \to 6} (x^2 - 3x - 15)$

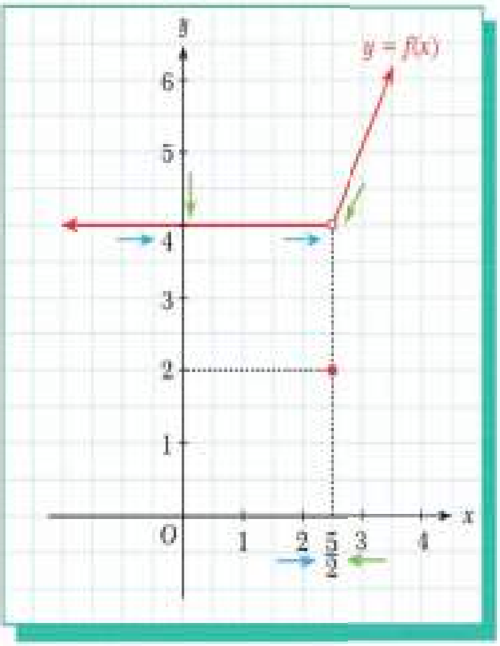
d. $\lim\limits_{x \to a} 2x(x + 1)$ e. $\lim\limits_{t \to -3} 3t^2(2t - 1)$ f. $\lim\limits_{x \to 3} 7$

Answers

2. a. 0 b. –3 c. 3 d. $2a^2 + 2a$ e. –189 f. 7

EXAMPLE 2 Given the piecewise function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} 4 & \text{if } x < \frac{5}{2} \\ 2 & \text{if } x = \frac{5}{2} \\ 2x - 1 & \text{if } x > \frac{5}{2} \end{cases}$, find $\lim\limits_{x \to \frac{5}{2}} f(x)$.

Solution Let us first draw the graph of the function. As we can see, $x = \frac{5}{2}$ is a crucial point in the graph. When we approach $\frac{5}{2}$ from the right-hand side (i.e. when x is greater than $\frac{5}{2}$) we use the function $f(x) = 2x - 1$ and get $\lim\limits_{x \to \frac{5}{2}} (2x - 1) = 4$. So we can say that f approaches 4 from the right-hand side. When we approach x from the left-hand side (i.e. when x is less than $\frac{5}{2}$) we use the function $f(x) = 4$, which is constant. Its limit is 4 when x approaches from the left-hand side. As a result, f approaches 4 as x approaches $\frac{5}{2}$ from both sides, i.e. $\lim\limits_{x \to \frac{5}{2}} f(x) = 4$.



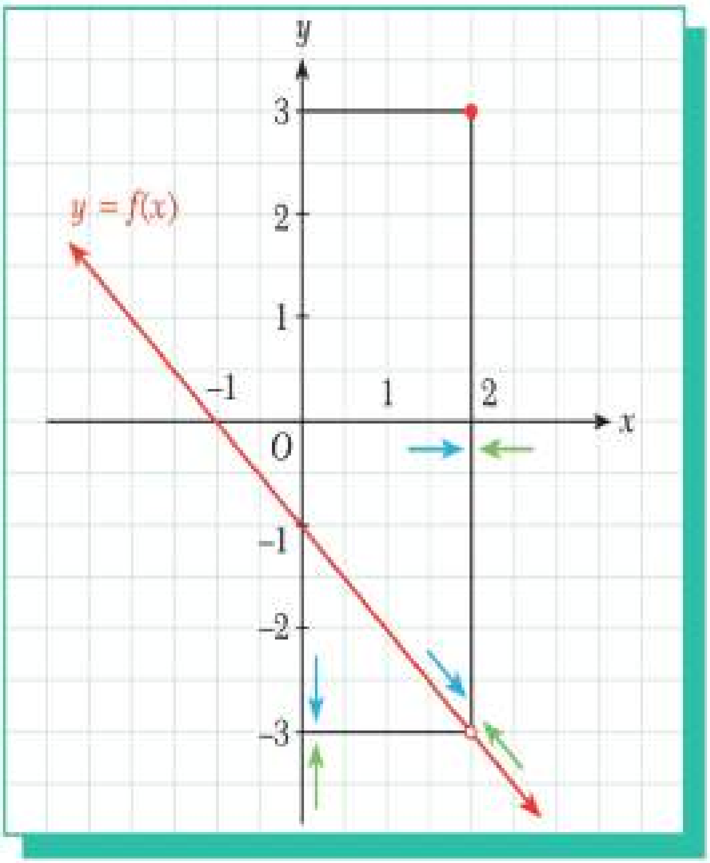
A point at which we need to check the right-hand and the left-hand limits of a function is called a **crucial point** of the function.

EXAMPLE 3 Given the piecewise function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} -x-1 & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}$, find $\lim\limits_{x \to 2} f(x)$.

Solution Let us draw the graph of $f(x)$.

$x = 2$ is a crucial point. Notice that $f(2) = 3$ but 3 is not the limit of f at 2. This is because the limit is the value which $f(x)$ approaches as x approaches 2. And in the graph we can see that the limit of $f(x)$ when x approaches 2 is –3.

As x gets closer to a point c, although the limit exists and approaches a number, at the point c a function may have a different value, or may not even be defined. What happens at the given point is not important for the limit at this point.



EXAMPLE 4 A piecewise function $f(x)$ is given as $f: \mathbb{R} \to \mathbb{R}, f(x) = \begin{cases} x^2 - 3 & \text{if} \quad x < -2 \\ 2 & \text{if} \quad x = -2. \\ x + 5 & \text{if} \quad x > -2 \end{cases}$

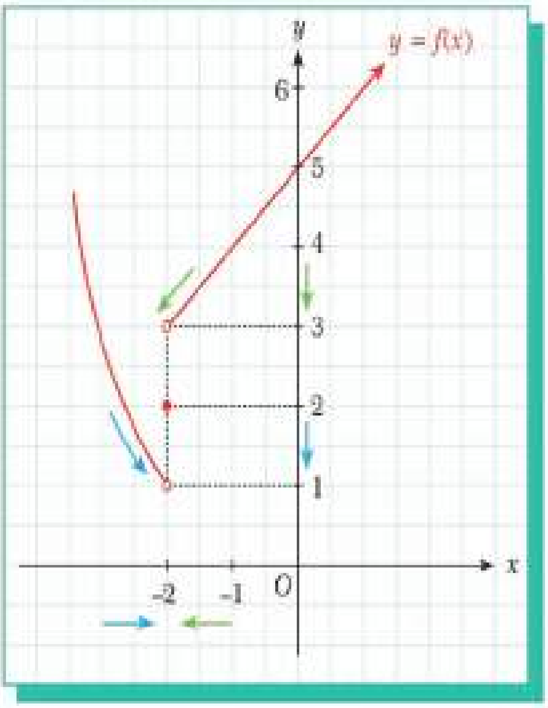
Find $\lim\limits_{x \to -2} f(x)$.

Solution First we draw a graph of the function. Notice that $x = -2$ is a crucial point. We need to examine what happens at the point $x = -2$ when we approach it from the left and from the right.

When we approach -2 from the right-hand side, the function is $f(x) = x + 5$ and $\lim\limits_{x \to -2} x + 5 = 3$, so f approaches 3.

On the other hand, when we approach -2 from the left-hand side, x is less than -2 and we use $f(x) = (x^2 - 3)$ so $\lim\limits_{x \to -2}(x^2 - 3) = 1$.

We can see that we get different results if we approach $x = -2$ from different sides. For this reason, we say that at this point the limit does not exist.



Note

If we get different results for a limit when we approach it from the right and from the left, we say that the limit does not exist at this point.

Check Yourself

Graph each function and evaluate the given limit.

1. $f: R \to R$, $f(x) = 3$, $\lim_{x \to 2} f(x)$

2. $f: R \to R$, $f(x) = -x - 2$, $\lim_{x \to 3} f(x)$

3. $f: R \to R$, $f(x) = \begin{cases} x^2 & \text{if } x \neq -2 \\ 1 & \text{if } x = -2 \end{cases}$, $\lim_{x \to -2} f(x)$

4. $f: R \to R$, $f(x) = \begin{cases} 1 & \text{if } x = 1, \\ x^2 + 1 & \text{if } x < 1 \end{cases}$ $\lim_{x \to 1} f(x)$

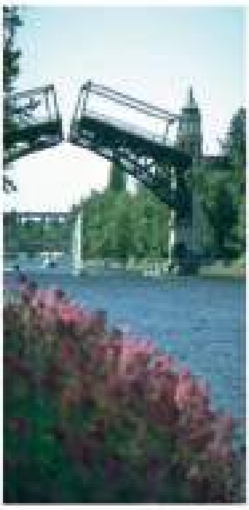
Answers

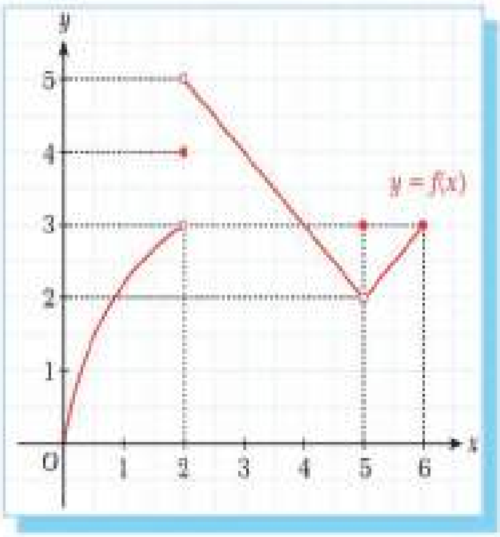
1. 3 2. -5 3. 4 4. does not exist

B. ONE-SIDED LIMITS

As we have already seen, sometimes the limit of a function can have two different values: one value when x approaches x_0 from the right, and another when x approaches x_0 from the left. When this happens, we call the limit of f as x approaches x_0 from the right the **right-hand limit of f at x_0** and write it as $\lim_{x \to x_0^+} f(x)$.

We call the limit of f as x approaches x_0 from the left the **left-hand limit of f at x_0** and write it as $\lim_{x \to x_0^-} f(x)$.





For example, consider the function $y = f(x)$ shown opposite. Let us find the left-hand and right-hand limits at the points 2, 5 and 6:

a. $\lim_{x \to 2^-} f(x) = 3$ and $\lim_{x \to 2^+} f(x) = 5$

b. $\lim_{x \to 5^-} f(x) = 2$ and $\lim_{x \to 5^+} f(x) = 2$

c. $\lim_{x \to 6^-} f(x) = 3$ and $\lim_{x \to 6^+} f(x)$ does not exist,

since f is not defined for $x > 6$.

In other words the left-hand and right-hand limits as $x \to 2$ are different; as $x \to 5$ the left- and right-hand limits are the same, and as $x \to 6$ only the left-hand limit exists.

Definition

existence of a limit

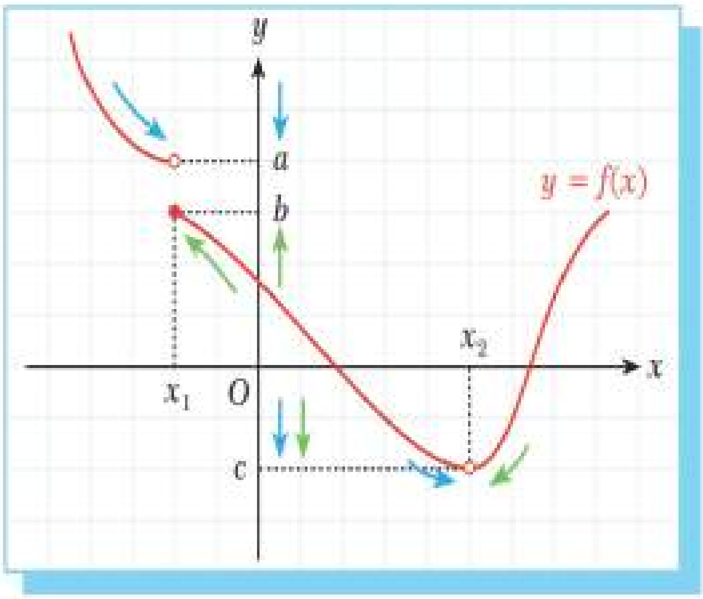
The limit of a function $f(x)$ at a point x_0 exists if and only if the right-hand and left-hand limits at x_0 exist and are equal.

In other words,

$$\lim_{x \to x_0} f(x) = L \Leftrightarrow \lim_{x \to x_0^+} f(x) = L \text{ and } \lim_{x \to x_0^-} f(x) = L.$$

The left-hand limit and right-hand limit of a function are also called the **one-sided limits** of the function.

For example, in the figure the function f has a limit at point x_2, but it has no limit at point x_1 because the left-hand and right-hand limits at x_1 are different.



EXAMPLE 5 $f: \mathbb{R} - \{2\} \to \mathbb{R}$. $f(x) = \begin{cases} x-1 & \text{if } x>2 \\ -x+3 & \text{if } x<2 \end{cases}$ is given. Find $\lim\limits_{x \to 2} f(x)$.

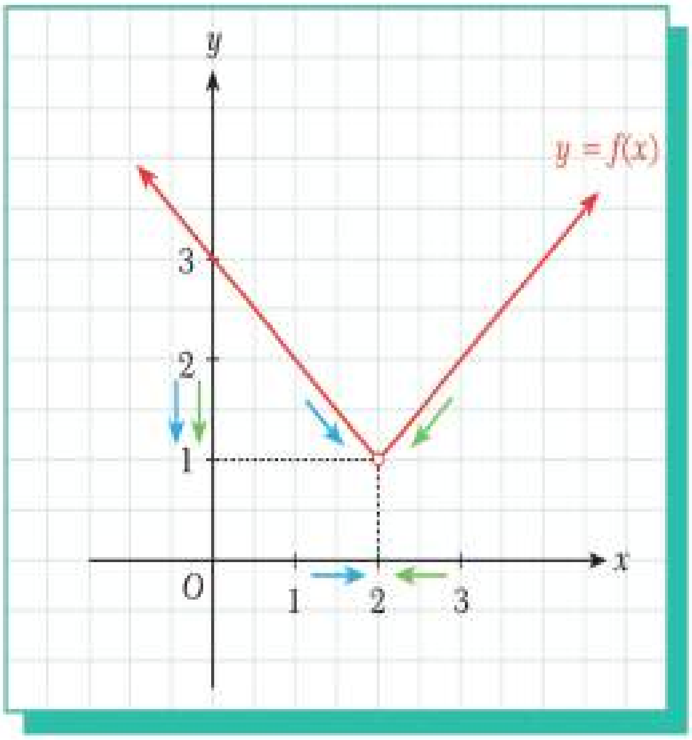
Solution As we can see from the graph, the point $x_0 = 2$ is the crucial point of $f(x)$.

Therefore, let us examine the one-sided limits at this point.

$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+}(x-1) = 2-1 = 1$

$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-}(-x+3) = -2+3 = 1$

Since $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^-} f(x) = 1$, they exist and are equal, and so $\lim\limits_{x \to 2} f(x) = 1$.



C. LIMITS OF SPECIAL FUNCTIONS

We have now learnt the definition and basic concepts of the limit of a function, and studied one-sided limits. In this section we will look at the limit of some special functions: the absolute value function, the sign function and the floor function.

We know that at a given point, the limit of a function exists if the right-hand limit and the left-hand limit exist and are equal.

We can evaluate a limit of an absolute value, sign or floor function at a point by treating the function as a piecewise function and checking the one-sided limits at the point. If the two limits exist and are equal, then we can say that a limit exists at the given point.

EXAMPLE 6

$f: \mathbb{R} - \{2\} \to \mathbb{R}$, $f(x) = \dfrac{|x-2|}{x-2} + x + 3$ is given. Find $\lim\limits_{x \to 2^+} f(x)$ and $\lim\limits_{x \to 2^-} f(x)$.

Solution Since the function f involves the absolute value expression $|x-2|$, $x_0 = 2$ is a crucial point for f.

Let us begin by writing the function as a piecewise function.

If $x > 2$, $x - 2 > 0$ and so $|x-2| = x - 2$. Therefore

$$f(x) = \frac{|x-2|}{x-2} + x + 3 = \frac{x-2}{x-2} + x + 3 = 1 + x + 3 = x + 4.$$

Similarly, if $x < 2$,

$x - 2 < 0$ and $|x-2| = -(x-2)$ and so

$$f(x) = \frac{|x-2|}{x-2} + x + 3 = \frac{-(x-2)}{x-2} + x + 3$$

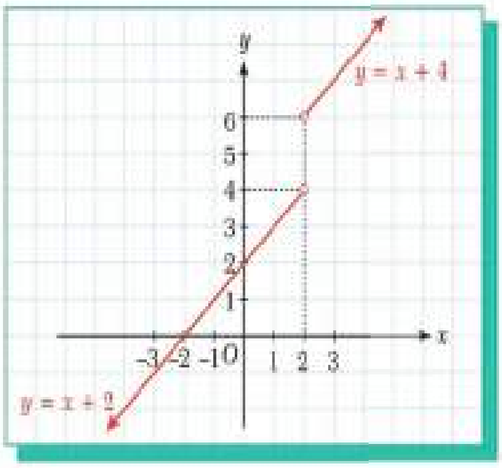
$$= -1 + x + 3 = x + 2.$$

In conclusion, $f(x) = \begin{cases} x+4 & \text{if } x > 2 \\ x+2 & \text{if } x < 2. \end{cases}$

So the limits are

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x+4) = 2 + 4 = 6$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x+2) = 2 + 2 = 4.$$



EXAMPLE 7 Find $\lim\limits_{x \to 1^-} \dfrac{|x-1|}{|1-x|}$.

Solution $x \to 1^-$ means x is less than 1. Therefore, $x - 1 < 0$ so $|x - 1| = -(x - 1) = 1 - x$, and $1 - x > 0$ so $|1 - x| = 1 - x$.

So $\lim\limits_{x \to 1^-} \dfrac{|x-1|}{|1-x|} = \lim\limits_{x \to 1^-} \dfrac{1-x}{1-x} = \lim\limits_{x \to 1^-} 1 = 1$.

Check Yourself

Evaluate the limits.

1. $\lim\limits_{x \to -3} (x^2 |x+1|)$
2. $\lim\limits_{x \to -1} \dfrac{x+|x|}{x}$
3. $\lim\limits_{x \to 2^-} \dfrac{|x^2-4|}{x-2}$
4. $\lim\limits_{x \to 1^-} \dfrac{|x^2-3x+2|}{x-1}$

Answers

1. 18 2. 0 3. –4 4. –1

Calculating Limits Using the Limit Laws

Historical Essay*

Limits

we used graphs and calculators to guess the values of limits. This section presents theorems for calculating limits and find limits of polynomials, rational functions, and powers. The fourth and fifth prepare for calculations later in the text.

The Limit Laws

The next theorem tells how to calculate limits of functions that are arithmetic combinations of functions whose limits we already know.

THEOREM 1 Limit Laws

If L, M, c and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule*: $\lim_{x \to c}(f(x) + g(x)) = L + M$

The limit of the sum of two functions is the sum of their limits.

2. *Difference Rule*: $\lim_{x \to c}(f(x) - g(x)) = L - M$

The limit of the difference of two functions is the difference of their limits.

3. *Product Rule*: $\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$

The limit of a product of two functions is the product of their limits.

4. *Constant Multiple Rule:* $\lim_{x \to c} (k \cdot f(x)) = k \cdot L$

The limit of a constant times a function is the constant times the limit of the function.

5. *Quotient Rule:* $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$

The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

6. *Power Rule:* If r and s are integers with no common factor and $s \neq 0$, then

$$\lim_{x \to c} (f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

The limit of a rational power of a function is that power of the limit of the function, provided the latter is a real number.

It is easy to convince ourselves that the properties in Theorem 1 are true (although these intuitive arguments do not constitute proofs). If x is sufficiently close to c, then $f(x)$ is close to L and $g(x)$ is close to M, from our informal definition of a limit. It is then reasonable that $f(x) + g(x)$ is close to $L + M$; $f(x) - g(x)$ is close to $L - M$; $f(x)g(x)$ is close to LM; $kf(x)$ is close to kL; and that $f(x)/g(x)$ is close to L/M if M is not zero. We prove the Sum Rule in Section 2.3, based on a precise definition of limit. Rules 2–5 are proved in Appendix 2. Rule 6 is proved in more advanced texts.

Here are some examples of how Theorem 1 can be used to find limits of polynomial and rational functions.

EXAMPLE 1 Using the Limit Laws

Use the observations $\lim_{x \to c} k = k$ and $\lim_{x \to c} x = c$ and the properties of limits to find the following limits.

(a) $\lim_{x \to c} (x^3 + 4x^2 - 3)$ **(b)** $\lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5}$

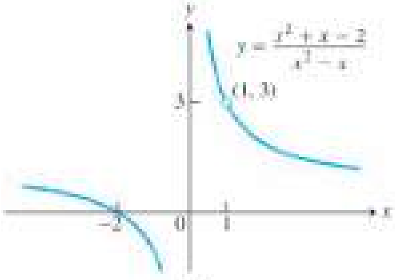
Solution

(a)
$$\begin{aligned}\lim_{x \to c} (x^3 + 4x^2 - 3) &= \lim_{x \to c} x^3 + \lim_{x \to c} 4x^2 - \lim_{x \to c} 3 && \text{Sum and Difference Rules}\\ &= c^3 + 4c^2 - 3 && \text{Product and Multiple Rules}\end{aligned}$$

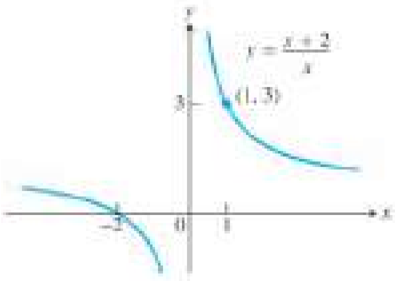
(b)
$$\begin{aligned}\lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5} &= \frac{\lim_{x \to c} (x^4 + x^2 - 1)}{\lim_{x \to c} (x^2 + 5)} && \text{Quotient Rule}\\ &= \frac{\lim_{x \to c} x^4 + \lim_{x \to c} x^2 - \lim_{x \to c} 1}{\lim_{x \to c} x^2 + \lim_{x \to c} 5} && \text{Sum and Difference Rules}\\ &= \frac{c^4 + c^2 - 1}{c^2 + 5} && \text{Power or Product Rule}\end{aligned}$$

EXAMPLE 2 Canceling a Common Factor

Evaluate $\lim_{x \to 1} \dfrac{x^2 + x - 2}{x^2 - x}$.



(a)



(b)

Solution We cannot substitute $x = 1$ because it makes the denominator zero. We test the numerator to see if it, too, is zero at $x = 1$. It is, so it has a factor of $(x - 1)$ in common with the denominator. Canceling the $(x - 1)$'s gives a simpler fraction with the same values as the original for $x \neq 1$:

$$\frac{x^2 + x - 2}{x^2 - x} = \frac{(x - 1)(x + 2)}{x(x - 1)} = \frac{x + 2}{x}, \qquad \text{if } x \neq 1.$$

Using the simpler fraction, we find the limit of these values as $x \to 1$ by substitution:

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \to 1} \frac{x + 2}{x} = \frac{1 + 2}{1} = 3.$$

See The Figure ■

EXAMPLE 3 Creating and Canceling a Common Factor

Evaluate $\lim_{x \to 0} \dfrac{\sqrt{x^2 + 100} - 10}{x^2}$.

Solution We cannot substitute $x = 0$, and the numerator and denominator have no obvious common factors.

We can create a common factor by multiplying both numerator and denominator by the expression $\sqrt{x^2 + 100} + 10$ (obtained by changing the sign after the square root). The preliminary algebra rationalizes the numerator:

$$\begin{aligned}
\frac{\sqrt{x^2 + 100} - 10}{x^2} &= \frac{\sqrt{x^2 + 100} - 10}{x^2} \cdot \frac{\sqrt{x^2 + 100} + 10}{\sqrt{x^2 + 100} + 10} \\
&= \frac{x^2 + 100 - 100}{x^2\left(\sqrt{x^2 + 100} + 10\right)} \\
&= \frac{x^2}{x^2\left(\sqrt{x^2 + 100} + 10\right)} && \text{Common factor } x^2 \\
&= \frac{1}{\sqrt{x^2 + 100} + 10}. && \text{Cancel } x^2 \text{ for } x \neq 0
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{x \to 0} \frac{\sqrt{x^2 + 100} - 10}{x^2} &= \lim_{x \to 0} \frac{1}{\sqrt{x^2 + 100} + 10} \\
&= \frac{1}{\sqrt{0^2 + 100} + 10} && \text{Denominator not 0 at } x = 0; \text{ substitute} \\
&= \frac{1}{20} = 0.05.
\end{aligned}$$

Limits Involving $(\sin \theta)/\theta$

A central fact about $(\sin \theta)/\theta$ is that in radian measure its limit as $\theta \to 0$ is 1. We can see this in Figure 5 -1 and confirm it algebraically using the Sandwich Theorem.

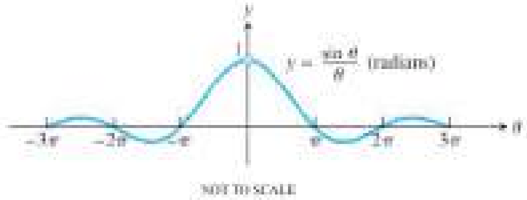


FIGURE 5 -1 The graph of $f(\theta) = (\sin \theta)/\theta$.

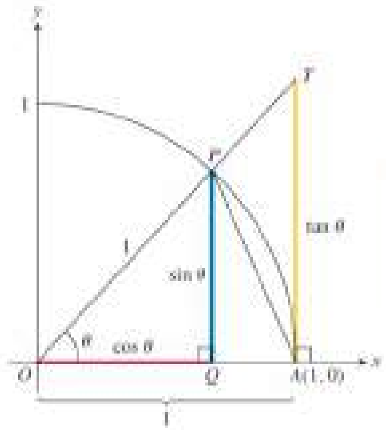


FIGURE 5 -1 The figure for the proof of Theorem 7. $TA/OA = \tan \theta$, but $OA = 1$, so $TA = \tan \theta$.

> Equation (2) is where radian measure comes in: The area of sector OAP is $\theta/2$ only if θ is measured in radians.

THEOREM

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \qquad (\theta \text{ in radians}) \qquad (1)$$

Proof The plan is to show that the right-hand and left-hand limits are both 1. Then we will know that the two-sided limit is 1 as well.

To show that the right-hand limit is 1, we begin with positive values of θ less than $\pi/2$ (Figure 5 -1). Notice that

$$\text{Area } \Delta OAP < \text{ area sector } OAP < \text{ area } \Delta OAT.$$

We can express these areas in terms of θ as follows:

$$\text{Area } \Delta OAP = \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}(1)(\sin \theta) = \frac{1}{2}\sin \theta$$

$$\text{Area sector } OAP = \frac{1}{2}r^2\theta = \frac{1}{2}(1)^2\theta = \frac{\theta}{2} \qquad (2)$$

$$\text{Area } \Delta OAT = \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}(1)(\tan \theta) = \frac{1}{2}\tan \theta.$$

Thus,

$$\frac{1}{2}\sin \theta < \frac{1}{2}\theta < \frac{1}{2}\tan \theta.$$

This last inequality goes the same way if we divide all three terms by the number $(1/2)\sin \theta$, which is positive since $0 < \theta < \pi/2$:

$$1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta}.$$

Taking reciprocals reverses the inequalities:

$$1 > \frac{\sin \theta}{\theta} > \cos \theta.$$

Since $\lim_{\theta \to 0^+} \cos \theta = 1$ the Sandwich Theorem gives $\lim_{\theta \to 0^+} \frac{\sin \theta}{\theta} = 1$.

Recall that $\sin\theta$ and θ are both *odd functions*. Therefore, $f(\theta) = (\sin\theta)/\theta$ is an *even function*, with a graph symmetric about the y-axis (see Figure 2.29). This symmetry implies that the left-hand limit at 0 exists and has the same value as the right-hand limit:

$$\lim_{\theta\to 0^-} \frac{\sin\theta}{\theta} = 1 = \lim_{\theta\to 0^+} \frac{\sin\theta}{\theta},$$

so $\lim_{\theta\to 0} (\sin\theta)/\theta = 1$ by Theorem 1.

EXAMPLE 4 Using $\lim_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$

Show that **(a)** $\lim_{h\to 0} \dfrac{\cos h - 1}{h} = 0$ and **(b)** $\lim_{x\to 0} \dfrac{\sin 2x}{5x} = \dfrac{2}{5}$.

Solution

(a) Using the half-angle formula $\cos h = 1 - 2\sin^2(h/2)$, we calculate

$$\begin{aligned}\lim_{h\to 0} \frac{\cos h - 1}{h} &= \lim_{h\to 0} -\frac{2\sin^2(h/2)}{h} \\ &= -\lim_{\theta\to 0} \frac{\sin\theta}{\theta}\sin\theta \qquad \text{Let } \theta = h/2. \\ &= -(1)(0) = 0.\end{aligned}$$

(b) Equation (1) does not apply to the original fraction. We need a $2x$ in the denominator, not a $5x$. We produce it by multiplying numerator and denominator by $2/5$:

$$\begin{aligned}\lim_{x\to 0} \frac{\sin 2x}{5x} &= \lim_{x\to 0} \frac{(2/5)\cdot \sin 2x}{(2/5)\cdot 5x} \\ &= \frac{2}{5}\lim_{x\to 0} \frac{\sin 2x}{2x} \qquad \text{Now, Eq. (1) applies with } \theta = 2x. \\ &= \frac{2}{5}(1) = \frac{2}{5}\end{aligned}$$

Limits Involving Trigonometric Functions

The trigonometric functions have important limit properties:

$$\lim_{x\to 0} sinx = 0$$

$$\lim_{x\to 0} \cos x = 1$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x\to 0} \frac{1-\cos x}{x} = 0$$

$$\lim_{x\to 0} tanx = 0$$

$$\lim_{x\to 0} \frac{\tan ax}{ax} = 1$$

You can use these properties to evaluate many limit problems involving the six basic trigonometric functions.

Example 1: Evaluate $\lim_{x\to 0} \frac{\cos x}{\sin x - 3}$.

Substituting 0 for *x*, you find that cos *x* approaches 1 and sin *x* − 3 approaches −3; hence,

$$\lim_{x\to 0} \frac{\cos x}{\sin x - 3} = -\frac{1}{3}$$

Example 2: Evaluate $\lim_{x\to 0^+} \cot x$.

Because cot *x* = cos *x*/sin *x*, you find $\lim_{x\to 0^+} \cos x/\sin x$. The numerator approaches 1 and the denominator approaches 0 through positive values because we are approaching 0 in the first quadrant; hence, the function increases without bound and $\lim_{x\to 0^+} \cot x = +\infty$, and the function has a vertical asymptote at *x* = 0.

Example 3: Evaluate $\lim_{x\to 0}\frac{\sin 4x}{x}$.

Multiplying the numerator and the denominator by 4 produces

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 4x}{x} &= \lim_{x\to 0}\frac{4\sin 4x}{4x}\\ &= \left(\lim_{x\to 0} 4\right)\cdot \lim_{x\to 0}\frac{\sin 4x}{4x}\\ &= 4\cdot 1\\ \lim_{x\to 0}\frac{\sin 4x}{x} &= 4\end{aligned}$$

Example 4: Evaluate $\lim_{x\to 0}\frac{\sec x - 1}{x}$.

Because sec x = 1/cos x, you find that

$$\begin{aligned}\lim_{x\to 0}\frac{\sec x - 1}{x} &= \lim_{x\to 0}\frac{\frac{1}{\cos x}-1}{x}\\ &= \lim_{x\to 0}\frac{1-\cos x}{x\cos x}\\ &= \lim_{x\to 0}\left(\frac{1}{\cos x}\right)\cdot\left(\frac{1-\cos x}{x}\right)\\ &= \left[\lim_{x\to 0}\frac{1}{\cos x}\right]\cdot\left[\lim_{x\to 0}\frac{1-\cos x}{x}\right]\\ &= 1\cdot 0\\ \lim_{x\to 0}\frac{\sec x - 1}{x} &= 0\end{aligned}$$

Exercises:

Evaluate:

1. $\lim_{x\to 0}\frac{\sin^2 4x}{x\tan 2x}$
2. $\lim_{x\to 0}\frac{\tan 4x+\tan 3x}{\sin 5x}$
3. $\lim_{x\to}[\frac{3x}{\sin 2x}+\frac{1-\cos 6x}{\sin^2 x}]$
4. $\lim_{x\to 0}[\sin 2x+\frac{\tan 4x}{6x}]$

Continuity

Continuity at a Point

To understand continuity, we need to consider a function like the one in Figure 5 -3 whose limits we investigated in Example 2.

FIGURE 5- 3 The function is continuous on [0, 4] except at $x = 1$, $x = 2$, and $x = 4$ (Example 1).

EXAMPLE 1 Investigating Continuity

Find the points at which the function f in Figure 2.50 is continuous and the points at which f is discontinuous.

Solution The function f is continuous at every point in its domain $[0, 4]$ except at $x = 1$, $x = 2$, and $x = 4$. At these points, there are breaks in the graph. Note the relationship between the limit of f and the value of f at each point of the function's domain.

Points at which f is continuous:

At $x = 0$, $\lim_{x \to 0^+} f(x) = f(0)$.

At $x = 3$, $\lim_{x \to 3} f(x) = f(3)$.

At $0 < c < 4, c \neq 1, 2$, $\lim_{x \to c} f(x) = f(c)$.

Points at which f is discontinuous:

At $x = 1$, $\lim_{x \to 1} f(x)$ does not exist.

At $x = 2$, $\lim_{x \to 2} f(x) = 1$, but $1 \neq f(2)$.

At $x = 4$, $\lim_{x \to 4^-} f(x) = 1$, but $1 \neq f(4)$.

At $c < 0, c > 4$, these points are not in the domain of f.

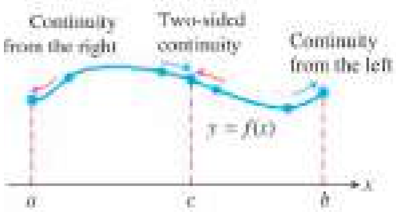


FIGURE 5- 4 Continuity at points a, b, and c.

To define continuity at a point in a function's domain, we need to define continuity at an interior point (which involves a two-sided limit) and continuity at an endpoint (which involves a one-sided limit) (Figure 5 - 4).

> **DEFINITION Continuous at a Point**
>
> *Interior point:* A function $y = f(x)$ is **continuous at an interior point c** of its domain if
>
> $$\lim_{x \to c} f(x) = f(c).$$
>
> *Endpoint:* A function $y = f(x)$ is **continuous at a left endpoint a** or is **continuous at a right endpoint b** of its domain if
>
> $$\lim_{x \to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively}.$$

If a function f is not continuous at a point c, we say that f is **discontinuous** at c and c is a **point of discontinuity** of f. Note that c need not be in the domain of f.

A function f is **right-continuous (continuous from the right)** at a point $x = c$ in its domain if $\lim_{x \to c^+} f(x) = f(c)$. It is **left-continuous (continuous from the left)** at c if $\lim_{x \to c^-} f(x) = f(c)$. Thus, a function is continuous at a left endpoint a of its domain if it is right-continuous at a and continuous at a right endpoint b of its domain if it is left-continuous at b. A function is continuous at an interior point c of its domain if and only if it is both right-continuous and left-continuous at c (Figure 5-4).

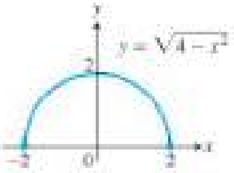


FIGURE 5-5 A function that is continuous at every domain point (Example 2).

EXAMPLE 2 A Function Continuous Throughout Its Domain

The function $f(x) = \sqrt{4 - x^2}$ is continuous at every point of its domain, $[-2, 2]$ (Figure 5-5), including $x = -2$, where f is right-continuous, and $x = 2$, where f is left-continuous. ■

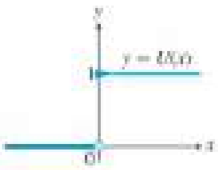


FIGURE 5-6 A function that is right-continuous, but not left-continuous, at the origin. It has a jump discontinuity there (Example 3).

EXAMPLE 3 The Unit Step Function Has a Jump Discontinuity

The unit step function $U(x)$, graphed in Figure 5-6, is right-continuous at $x = 0$, but is neither left-continuous nor continuous there. It has a jump discontinuity at $x = 0$. ■

We summarize continuity at a point in the form of a test.

> **Continuity Test**
>
> A function $f(x)$ is continuous at $x = c$ if and only if it meets the following three conditions.
>
> 1. $f(c)$ exists (c lies in the domain of f)
> 2. $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$)
> 3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value)

For one-sided continuity and continuity at an endpoint, the limits in parts 2 and 3 of the test should be replaced by the appropriate one-sided limits.

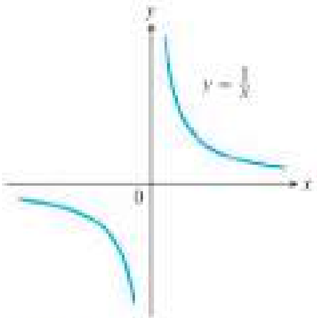


FIGURE 5 - 7 The function $y = 1/x$ is continuous at every value of x except $x = 0$. It has a point of discontinuity at $x = 0$ (Example 5).

Continuous Functions

A function is **continuous on an interval** if and only if it is continuous at every point of the interval. For example, the semicircle function graphed in Figure 5-5 is continuous on the interval $[-2, 2]$, which is its domain. A **continuous function** is one that is continuous at every point of its domain. A continuous function need not be continuous on every interval. For example, $y = 1/x$ is not continuous on $[-1, 1]$ (Figure 5 - 7), but it is continuous over its domain $(-\infty, 0) \cup (0, \infty)$.

EXAMPLE 4 Identifying Continuous Functions

(a) The function $y = 1/x$ (Figure 5 - 7) is a continuous function because it is continuous at every point of its domain. It has a point of discontinuity at $x = 0$, however, because it is not defined there.

(b) The identity function $f(x) = x$ and constant functions are continuous everywhere by Example 3.

Algebraic combinations of continuous functions are continuous wherever they are defined.

> **THEOREM 3 Properties of Continuous Functions**
>
> If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.
>
> 1. *Sums:* $f + g$
> 2. *Differences:* $f - g$
> 3. *Products:* $f \cdot g$
> 4. *Constant multiples:* $k \cdot f$, for any number k
> 5. *Quotients:* f/g provided $g(c) \neq 0$
> 6. *Powers:* $f^{r/s}$, provided it is defined on an open interval containing c, where r and s are integers

Most of the results in Theorem 3 are easily proved from the limit rules in Theorem 1, Section 2.2. For instance, to prove the sum property we have

$$\begin{aligned}
\lim_{x\to c}(f + g)(x) &= \lim_{x\to c}(f(x) + g(x)) \\
&= \lim_{x\to c} f(x) + \lim_{x\to c} g(x), && \text{Sum Rule, Theorem 1} \\
&= f(c) + g(c) && \text{Continuity of } f, g \text{ at } c \\
&= (f + g)(c).
\end{aligned}$$

This shows that $f + g$ is continuous.

EXAMPLE 5 Polynomial and Rational Functions Are Continuous

(a) Every polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is continuous because $\lim_{x\to c} P(x) = P(c)$ by Theorem 2, Section 2.2.

(b) If $P(x)$ and $Q(x)$ are polynomials, then the rational function $P(x)/Q(x)$ is continuous wherever it is defined ($Q(c) \neq 0$) by the Quotient Rule in Theorem 9.

EXAMPLE 6 Continuity of the Absolute Value Function

The function $f(x) = |x|$ is continuous at every value of x. If $x > 0$, we have $f(x) = x$, a polynomial. If $x < 0$, we have $f(x) = -x$, another polynomial. Finally, at the origin, $\lim_{x\to 0}|x| = 0 = |0|$.

The functions $y = \sin x$ and $y = \cos x$ are continuous at $x = 0$ by Example 6 of Both functions are, in fact, continuous everywhere . It follows from Theorem 3 that all six trigonometric functions are then continuous wherever they are defined. For example, $y = \tan x$ is continuous on $\cdots \cup (-\pi/2, \pi/2) \cup (\pi/2, 3\pi/2) \cup \cdots$.

Continuous Extensions

1. Define $g(3)$ in a way that extends $g(x) = (x^2 - 9)/(x - 3)$ to be continuous at $x = 3$.
2. Define $h(2)$ in a way that extends $h(t) = (t^2 + 3t - 10)/(t - 2)$ to be continuous at $t = 2$.
3. Define $f(1)$ in a way that extends $f(s) = (s^3 - 1)/(s^2 - 1)$ to be continuous at $s = 1$.
4. Define $g(4)$ in a way that extends $g(x) = (x^2 - 16)/(x^2 - 3x - 4)$ to be continuous at $x = 4$.
5. For what value of a is

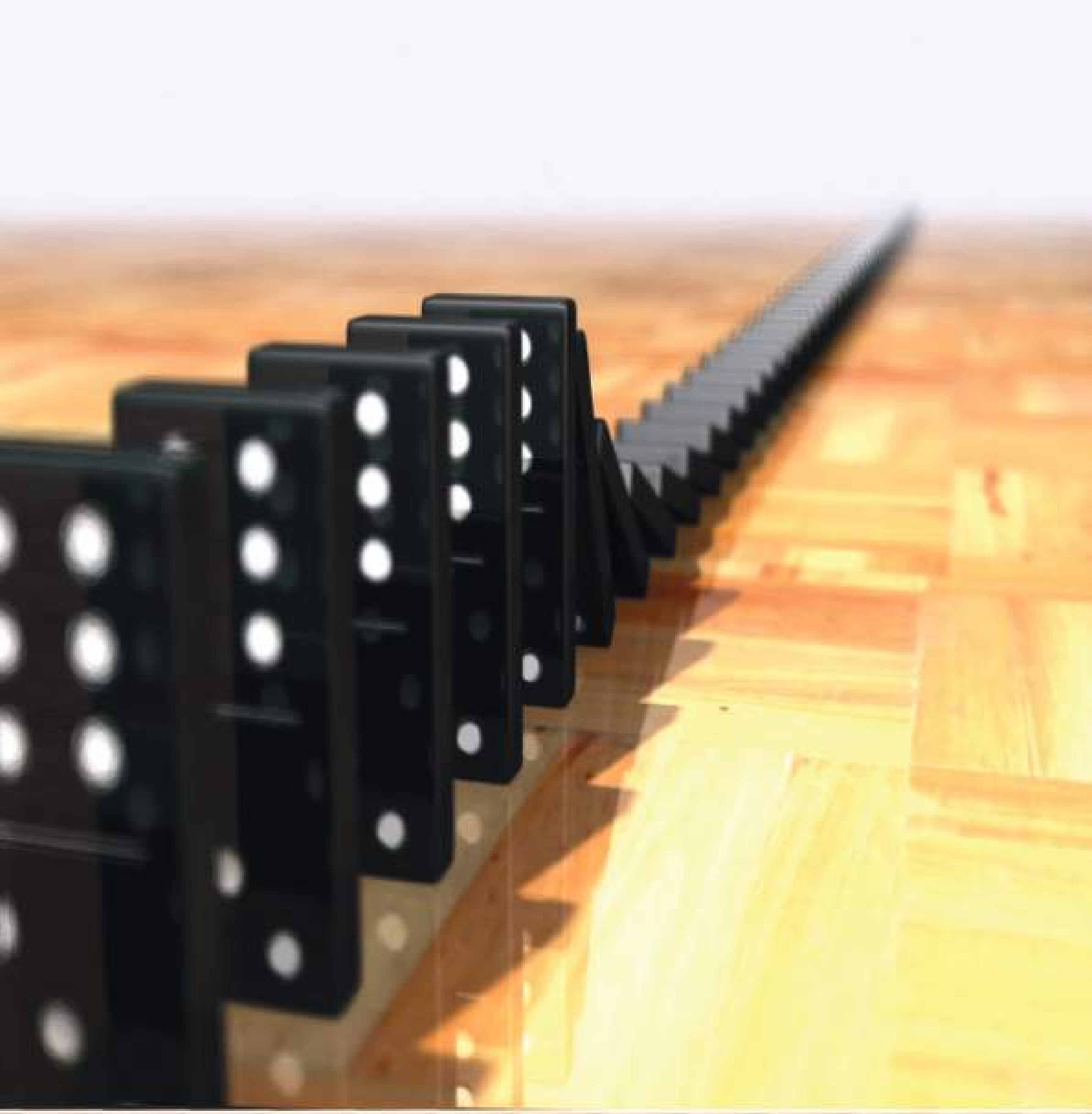
$$f(x) = \begin{cases} x^2 - 1, & x < 3 \\ 2ax, & x \geq 3 \end{cases}$$

continuous at every x?

6. For what value of b is

$$g(x) = \begin{cases} x, & x < -2 \\ bx^2, & x \geq -2 \end{cases}$$

continuous at every x?



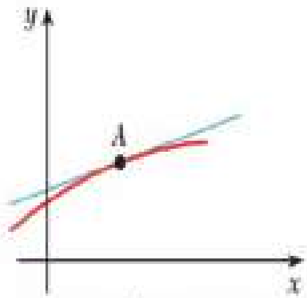
Chapter 6

DERIVATION

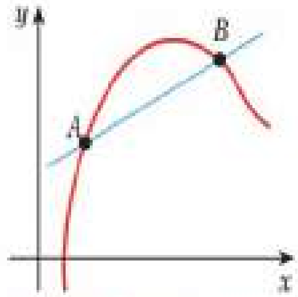
DERIVATION

A. TANGENTS

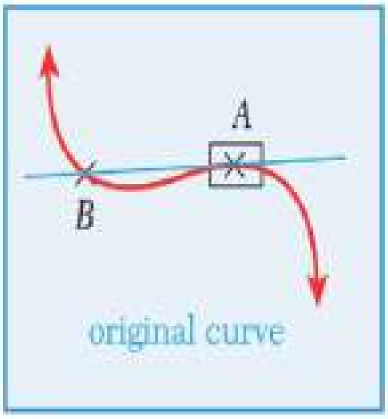
The word 'tangent' comes from the Latin word *tangens*, which means 'touching'. Thus, a tangent line to a curve is a line that "just touches" the curve. In other words, a tangent line should be parallel to the curve at the point of contact. How can we explain this idea clearly? Look at the figures below.

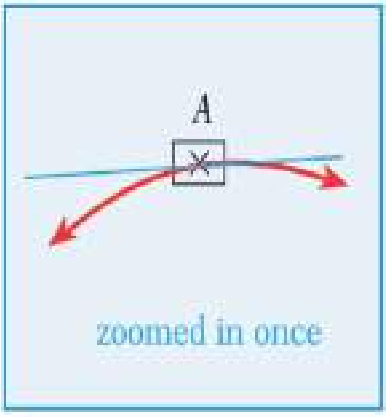


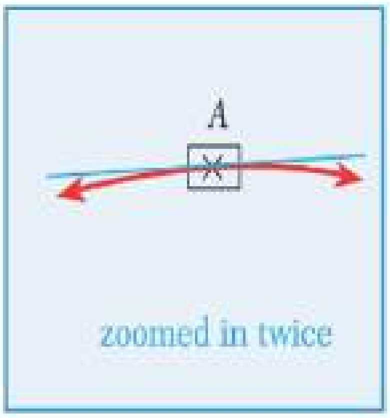
a tangent line to a curve



a secant line to a curve

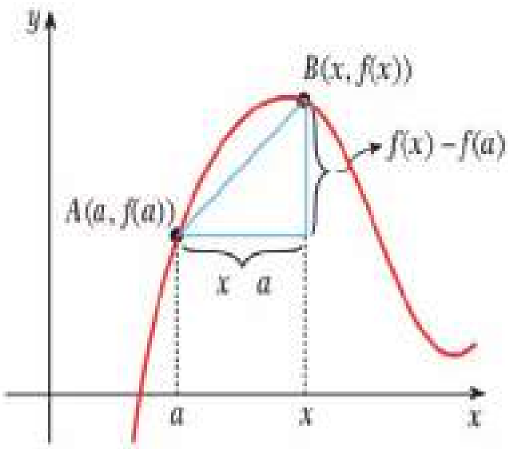


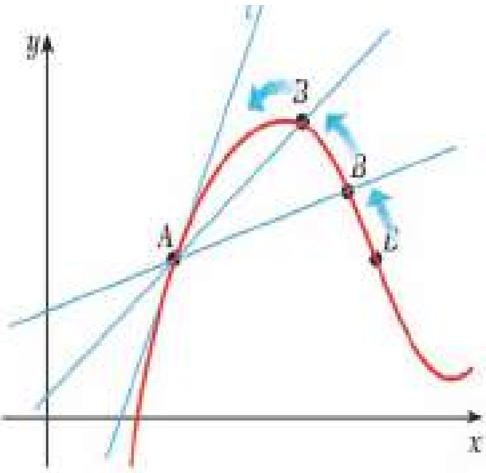




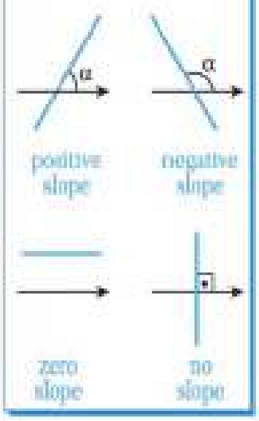
As we zoom in to the curve near the point A, the curve becomes almost indistinguishable from the tangent line. So, the tangent line is parallel to the curve at the point A.

How can we find the equation of a tangent to a curve at a given point? The graphs below show one approach.





The slope of the line is the tangent of the angle between the line and the positive x-axis.



The first graph shows the curve $y = f(x)$. The points $A(a, f(a))$ and $B(x, f(x))$ are two points on this curve. The secant line AB has slope m_{AB}, where

$$m_{AB} = \frac{f(x) - f(a)}{x - a}.$$

Now suppose that we want to find the slope of the tangent to the curve at point A. The second graph above shows what happens when we move point B closer and closer to point A on the curve. We can see that the slope of the secant line AB gets closer and closer to the slope of the tangent at A (line t). In other words, if m is the slope of the tangent line, then as B approches A, m_{AB} approaches m.

Definition

tangent line

The **tangent line** to the curve $y = f(x)$ at the point $A(a, f(a))$ is the line through A with the slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

provided that this limit exists.

Example 1 Find the equation of the tangent line to the curve $y = x^2$ at the point $A(1, 1)$.

Solution We can begin by calculating the slope of the tangent.

Here we have $a = 1$ and $f(x) = x^2$, so the slope is

> The equation of a line through the point (x_1, y_1) with slope m:
> $y - y_1 = m(x - x_1)$.

$$m = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

$$m = \lim_{x \to 1} \frac{(x-1)(x+1)}{(x-1)} = \lim_{x \to 1}(x+1) = 1 + 1 = 2.$$

Now we can write the equation of the tangent at point (1, 1):

$y - y_1 = m(x - x_1)$

$y - 1 = 2(x - 1)$

$y = 2x - 1.$

Example 2 Find the equation of the tangent line to the curve $y = x^3 - 1$ at the point $(-1, -2)$.

Solution Here we have $a = -1$ and $f(x) = x^3 - 1$, so the slope is

> $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

$$m = \lim_{x \to -1} \frac{f(x) - f(-1)}{x - (-1)} = \lim_{x \to -1} \frac{(x^3 - 1) - ((-1)^3 - 1)}{x + 1} = \lim_{x \to -1} \frac{x^3 + 1}{x + 1}$$

$$m = \lim_{x \to -1} \frac{(x+1)(x^2 - x + 1)}{(x+1)}$$

$$m = \lim_{x \to -1}(x^2 - x + 1) = (-1)^2 - (-1) + 1$$

$$m = 3.$$

So the equation of the tangent line at $(-1, -2)$ with slope $m = 3$ is

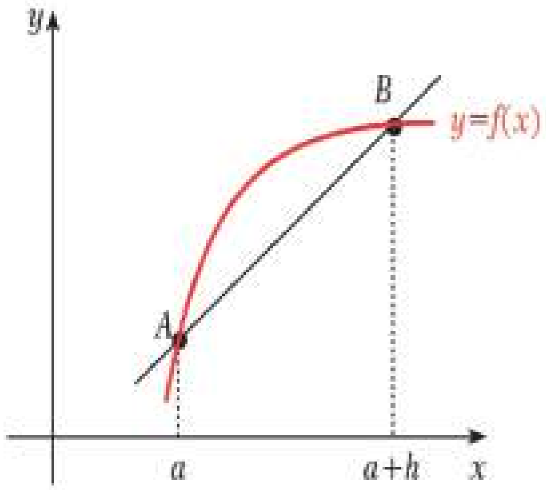
$y - y_1 = m(x - x_1)$

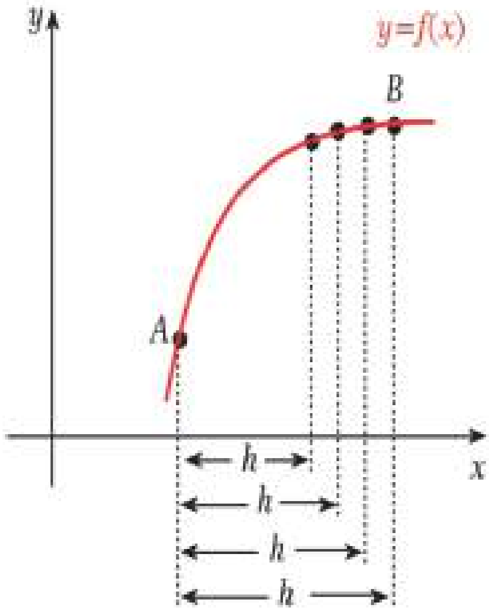
$y - (-2) = 3(x - (-1))$

$y + 2 = 3x + 3$

$y = 3x + 1.$

We can also write the expression for the slope of a tangent line in a different way. Look at the graphs below.





THE SLOPE OF A TANGENT LINE TO A CURVE

The slope of a tangent line to a curve $y = f(x)$ at $x = a$ is

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Example 3 Find the equation of the tangent line to the curve $y = x^3$ at the point $(-1, -1)$.

Solution Let $f(x) = x^3$. Then the slope of the tangent at $(-1, -1)$ is

$$m = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h \to 0} \frac{(-1+h)^3 - (-1)^3}{h}$$

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$$m = \lim_{h \to 0} \frac{(-1)^3 + 3(-1)^2h + 3(-1)h^2 + h^3 - (-1)^3}{h}$$

$$m = \lim_{h \to 0} \frac{h(3 - 3h + h^2)}{h} = \lim_{h \to 0}(3 - 3h + h^2) = 3.$$

So, the equation of the tangent at point $(-1, -1)$ is

$y - (-1) = 3(x - (-1))$

$y + 1 = 3x + 3$

$y = 3x + 2.$

Example 4 Find the equation of the normal line to the curve $y = \frac{2}{x}$ at the point (2, 1).

Solution Recall that a normal line is a line which is perpendicular to a tangent. The product of the slopes m_t of a tangent and m_n of a normal is –1.

Let us begin by finding the slope of the tangent.

$$m_t = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\frac{\frac{2}{2+h}-\frac{2}{2}}{h} = \lim_{h\to 0}\frac{\frac{2}{2+h}-1}{h}$$

$$m_t = \lim_{h\to 0}\frac{2-(2+h)}{h(h+2)} = \lim_{h\to 0}\frac{-h}{h(h+2)} = \lim_{h\to 0}\frac{-1}{2+h}$$

$$m_t = -\frac{1}{2}.$$

$m_n \cdot m_t = -1$
The product of slopes of the tangent line and the normal line at a point equals –1.

We have $m_t \cdot m_n = -1$.

So, $m_n = \frac{-1}{m_t} = \frac{-1}{-\frac{1}{2}} = 2.$

The equation of the normal line passing through the point (2, 1) with the slope $m_n = 2$ is

$y - y_1 = m_n(x - x_1)$

$y - 1 = 2(x - 2)$

$y = 2x - 3.$

Check Yourself

1. Find the equation of the tangent line to each curve at the given point P.
   a. $f(x) = x^2 - 1$ $P(-1, 0)$
   b. $f(x) = x^3 + 1$ $P(0, 1)$
   c. $f(x) = \frac{1}{x}$ $P(\frac{1}{2}, 2)$
2. Find the equation of the normal line at point P for each curve in the previous question.

Answers

1. a. $y = -2x - 2$ b. $y = 1$ c. $y = -4x + 4$

2. a. $y = \frac{1}{2}x + \frac{1}{2}$ b. $x = 0$ c. $y = \frac{1}{4}x + \frac{15}{8}$

B. DERIVATIVE OF A FUNCTION

Up to now we have treated the expression $\frac{f(x+h)-f(x)}{h}$ as a 'difference quotient' of the function $f(x)$. We have calculated the limit of a difference quotient as h approaches zero. Since this type of limit occurs so widely, it is given a special name and notation.

Definition

derivative of a function

The **derivative** of the function $f(x)$ with respect to x is the function $f'(x)$ (read as "f prime of x") defined by

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

The process of calculating the derivative is called **differentiation**. We say that $f(x)$ is **differentiable** at c if $f'(c)$ exists.

Thus, the derivative of a function $f(x)$ is the function $f'(x)$, which gives

1. the slope of the tangent line to the graph of $f(x)$ at any point $(x, f(x))$,

2. the rate of change of $f(x)$ at x.

FOUR-STEP PROCESS FOR FINDING f'(x)

1. Compute $f(x+h)$.
2. Form the difference $f(x+h)-f(x)$.
3. Form the quotient $\frac{f(x+h)-f(x)}{h}$.
4. Compute $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.

Example 5 Find the derivative of the function $f(x) = x^2$.

Solution To find $f'(x)$, we use the four-step process:

1. $f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$
2. $f(x+h) - f(x) = x^2 + 2xh + h^2 - x^2 = 2xh + h^2$
3. $\frac{f(x+h)-f(x)}{h}=\frac{2xh+h^2}{h}=\frac{h(2x+h)}{h}=2x+h$
4. $\lim_{h\to 0}(2x+h)=2x$

Thus, $f'(x) = 2x$.

Example 6 Find the derivative of the function $f(x) = x^2 - 8x + 9$ at $x = 1$.

Solution We apply the four-step process:

1. $f(x + h) = (x + h)^2 - 8(x + h) + 9 = x^2 + 2xh + h^2 - 8x - 8h + 9$
2. $f(x + h) - f(x) = x^2 + 2xh + h^2 - 8x - 8h + 9 - (x^2 - 8x + 9) = 2xh + h^2 - 8h$
3. $\dfrac{f(x+h)-f(x)}{h} = \dfrac{h^2+2xh-8h}{h} = h+2x-8$
4. $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0}(h+2x-8) = 2x-8$

So, $f'(x) = 2x - 8$ and $f'(1) = 2 \cdot 1 - 8 = -6$.

This result tells us that the slope of the tangent line to the graph of $f(x)$ at the point $x = 1$ is -6. It also tells us that the function $f(x)$ is changing at the rate of -6 units per unit change in x at $x = 1$.

Example 7 Let $f(x) = \dfrac{1}{x}$.

a. Find $f'(x)$.

b. Find the equation of the tangent line to the graph of $f(x)$ at the point $(1, 1)$.

Solution a. $f'(x) = \lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0}\dfrac{\frac{1}{x+h}-\frac{1}{x}}{h} = \lim\limits_{h\to 0}\dfrac{-\frac{h}{x(x+h)}}{h} = \lim\limits_{h\to 0}(-\dfrac{1}{x(x+h)}) = -\dfrac{1}{x^2}$.

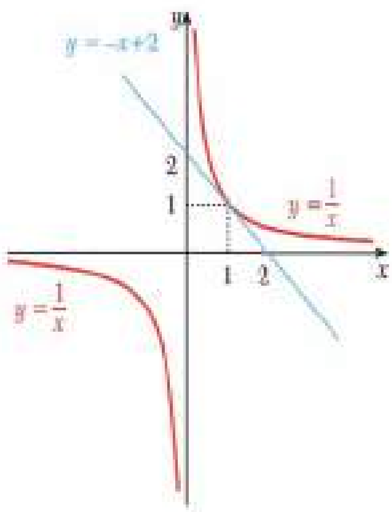
b. In order to find the equation of a tangent line, we have to find its slope and one point on the tangent line. We know that the derivative gives us the slope of the tangent. Let m be the slope of the tangent line, then

$$m = f'(1) = -\frac{1}{1^2} = -1.$$

So, the equation of the tangent line to the graph of $f(x)$ at the point $(1, 1)$ with the slope $m = -1$ is

$y - 1 = -1(x - 1)$

$y = -x + 2$.



Example 8 The function $f(x) = \sqrt{x}$ is given. Find the derivative of $f(x)$ and the equation of the normal line to $f(x)$ at the point $x = 1$.

Solution

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0}\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right)$$

$$f'(x) = \lim_{h\to 0}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

Remember that if m_t is the slope of a tangent and m_n is the slope of a normal at the same point, then $m_t \cdot m_n = -1$. So, we can find the slope of the normal from the slope of the tangent. Then we can write the equation of normal line to $f(x)$ at the point $x = 1$.

The slope of the tangent is

$$m_t = f'(1) = \frac{1}{2\cdot\sqrt{1}} = \frac{1}{2}.$$

The slope of the normal is

$$m_n = -\frac{1}{m_t} = -\frac{1}{\frac{1}{2}} = -2.$$

The equation of the normal line is

$y - y_0 = m_n \cdot (x - x_0)$

$y - 1 = -2(x - 1)$ *(Note that $y_0 = f(x_0)$, that is $y_0 = f(1) = 1$)*

$y = -2x + 3.$

Check Yourself

1. Find the derivative of the function $f(x) = 2x + 7$.
2. Let $f(x) = 2x^2 - 3x$.
   a. Find $f'(x)$.
   b. Find the equation of the tangent line to the graph of $f(x)$ at the point $x = 2$.
3. Find the derivative of the function $f(x) = x^3 - x$.
4. If $f(x) = \frac{1}{\sqrt{x+2}}$, find the derivative of $f(x)$.

Answers

1. 2 2. a. $4x - 3$ b. $y = 5x - 8$ 3. $3x^2 - 1$ 4. $-\frac{1}{2\sqrt{(x+2)^3}}$

C. LEFT-HAND AND RIGHT-HAND DERIVATIVES

When we were studying limits we learned that the limit of a function exists if and only if the left-hand and the right-hand limits exist and are equal. Otherwise the function has no limit. From this point, we may conclude that the derivative of a function $f(x)$ exists if and only if

$f'(x^-) = \lim\limits_{h \to 0^-} \dfrac{f(x+h) - f(x)}{h}$ and $f'(x^+) = \lim\limits_{h \to 0^+} \dfrac{f(x+h) - f(x)}{h}$ exist and are equal.

These expressions are respectively called the **left-hand derivative** and the **right-hand derivative** of the function.

Example 9 Show that the function $f(x) = \sqrt{x}$ does not have a derivative at the point $x = 0$.

Solution Here we should find the left-hand derivative and the right-hand derivative. If they exist, then we will check whether they are equal or not.

Let us find the left-hand derivative:

$$f'(0^-) = \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{\sqrt{0+h} - \sqrt{0}}{h} = \lim_{h \to 0^-} \frac{\sqrt{h}}{h}.$$

Since $h < 0$, $\sqrt{h}$ is undefined and this limit does not exist. So the left-hand derivative does not exist either.

Thus, the function $f(x) = \sqrt{x}$ has no derivative at the point $x = 0$.

Example 10 $f(x)$ is given as $f(x) = \begin{cases} x^2 - 1, & x \geq 1 \\ 2x - 2, & x < 1 \end{cases}$.

Does this function have a derivative at the point $x = 1$?

Solution We will find the left-hand and the right-hand derivatives.

$$f'(1^-) = \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{2(1+h) - 2 - 0}{h} = \lim_{h \to 0^-} \frac{2h}{h} = 2$$

$$f'(1^+) = \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{(1+h)^2 - 1 - 0}{h} = \lim_{h \to 0^+} \frac{h^2 + 2h}{h} = \lim_{h \to 0^+} (h+2) = 2.$$

The left-hand and the right-hand derivatives are equal to each other. Thus, the derivative of the function at the point $x = 1$ exists and

$$f'(1) = f'(1^-) = f'(1^+) = 2.$$

D. DIFFERENTIABILITY AND CONTINUITY

Recall that if $f'(c)$ exists, then the function $f(x)$ is differentiable at point c. Similarly, if $f(x)$ is differentiable on an open interval (a, b), then it is differentiable at every number in the interval (a, b).

Example 11 Where is the function $f(x) = |x|$ differentiable?

Solution We can approach this problem by testing the differentiability on three intervals: $x > 0$, $x < 0$ and $x = 0$

1. If $x > 0$, then $x + h > 0$ and $|x + h| = x + h$.

   Therefore, for $x > 0$ we have

   $$f'(x) = \lim_{h \to 0} \frac{|x+h| - |x|}{h} = \lim_{h \to 0} \frac{x+h-x}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1.$$

   So, $f'(x)$ exists and $f(x)$ is differentiable for any $x > 0$.

2. If $x < 0$, then $|x| = -x$ and $|x + h| = -(x + h)$ if we choose h small enough such that it is nearly equal to zero.

   Therefore, for $x < 0$ we have

   $$f'(x) = \lim_{x \to 0} \frac{|x+h| - |x|}{h} = \lim_{x \to 0} \frac{-(x+h)-(-x)}{h} = \lim_{h \to 0} \frac{-h}{h} = \lim_{h \to 0}(-1) = -1$$

   So, $f'(x)$ exists and $f(x)$ is differentiable for any $x < 0$.

3. For $x = 0$ we have to investigate the left-hand and the right-hand derivatives separately:

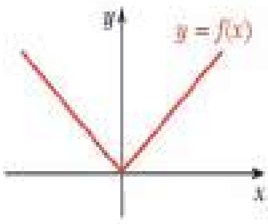
   $$\lim_{h \to 0^+} \frac{|0+h| - |0|}{h} = \lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1$$

   $$\lim_{h \to 0^-} \frac{|0+h| - |0|}{h} = \lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-}(-1) = -1$$

   Since these limits are different, $f'(x)$ does not exist. So, $f(x)$ is not differentiable for $x = 0$.

In conclusion, $f(x)$ is differentiable for all the values of x except 0.

Alternatively, from the graph of $f(x)$, we can see that $f(x)$ does not have a tangent line at the point $x = 0$. So, the derivative does not exist.



Note that the function does not have a derivative at the point where the graph has a 'corner'.

A function f is continuous at $x = a$ if and only if $\lim_{x \to a} f(x) = f(a)$.

If a function $f(x)$ is differentiable at a point, then its graph has a non-vertical tangent line at this point. It means that the graph of the function cannot have a 'hole' or 'gap' at this point. Thus, the function must be continuous at this point where it is differentiable.

Note

If $f(x)$ is differentiable at a, then $f(x)$ is continuous at a.

The converse, however, is not true: a continuous function may not be differentiable at every point.

For example, the function $f(x) = |x|$ is continuous at 0, because $\lim_{x \to 0} f(x) = 0 = f(0)$.

But it is not differentiable at the point $x = 0$.

Example 12 The piecewise function $f(x)$ is given as $f(x) = \begin{cases} 2x^2 - x, & x > 2 \\ 6, & x = 2. \\ x^3 - 2, & x < 2 \end{cases}$

a. Is $f(x)$ continuous at $x = 2$?

b. Is $f(x)$ differentiable at $x = 2$?

Solution a. Since $\lim_{x \to 2} f(x) = f(2)$, $f(x)$ is continuous at $x = 2$.

b. Let us find the left-hand and the right-hand derivatives of the function $f(x)$ at the point $x = 2$.

$$f(2^-) = \lim_{h \to 0^-} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0^-} \frac{(2+h)^3 - 2 - 6}{h}$$

$$= \lim_{h \to 0^-} \frac{2^3 + 3 \cdot 2^2 h + 3 \cdot 2h^2 + h^3 - 8}{h} = \lim_{h \to 0^-} \frac{h(12 + 6h + h^2)}{h}$$

$$= \lim_{h \to 0^-} (12 + 6h + h^2) = 12$$

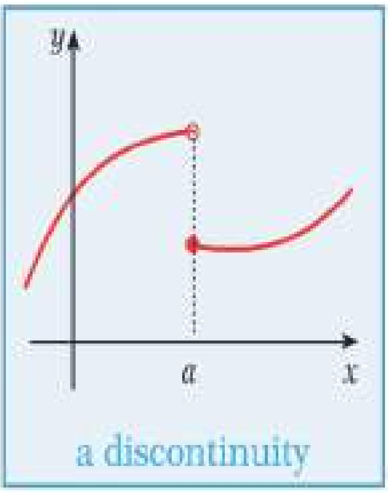
$$f(2^+) = \lim_{h \to 0^+} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0^+} \frac{2 \cdot (2+h)^2 - (2+h) - 6}{h}$$

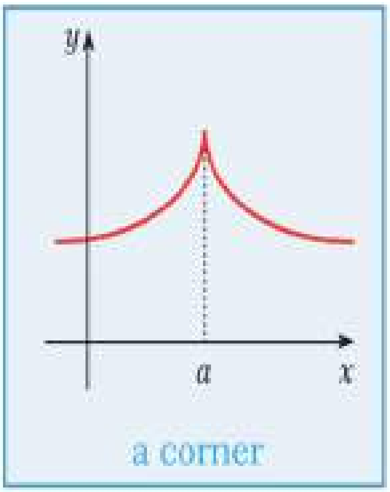
$$= \lim_{h \to 0^+} \frac{2 \cdot (4 + 4h + h^2) - 2 - h - 6}{h} = \lim_{h \to 0^+} \frac{8 + 8h + 2h^2 - 2 - h - 6}{h}$$

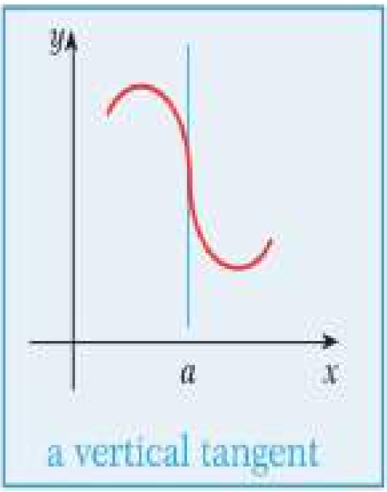
$$= \lim_{h \to 0^+} \frac{h(7 + 2h)}{h} = \lim_{h \to 0^+} (7 + 2h) = 7.$$

Since $f'(2^+) \neq f'(2^-)$, the derivative of the function $f(x)$ does not exist at the point $x = 2$. So, the function is continuous at $x = 2$, but it is not differentiable at the same point.

We have seen that a function $f(x)$ is not differentiable at a point if its graph is not continuous at $x = a$. The figures below show two more cases in which $f(x)$ is not differentiable at $x = a$:







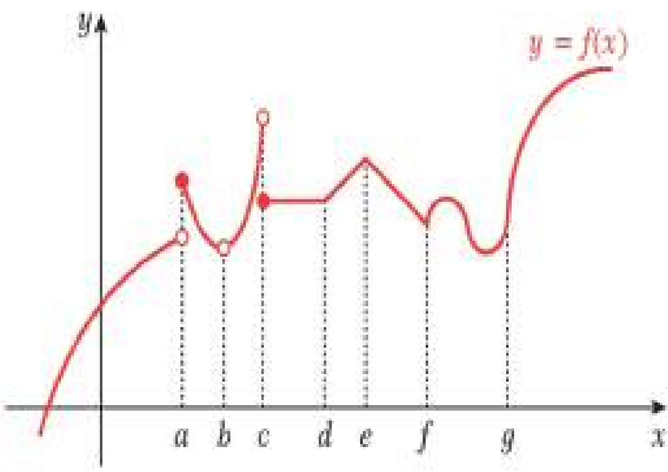
CRITERIA FOR DIFFERENTIABILITY

For the following cases the function is not differentiable at a given point:

1. the graph has a discontinuity at the point,
2. the graph has a 'corner' at the point,
3. the graph has a vertical tangent line at the point.

Example 13 Explain why the function shown in the graph givin below is not differentiable at each of the points $x = a, b, c, d, e, f, g$.

Solution The function $f(x)$ is not differentiable at the points $x = a, b, c$ because it is discontinuous at each of these points. The derivative of the function $f(x)$ does not exist at $x = d, e, f$ because it has a corner at each of these points. Finally, the function is not differentiable at $x = g$ because the tangent line is vertical at that point.



Check Yourself

1. Given that $f(x)=\begin{cases} x-1, & x<1 \\ x^2-1, & x\geq 1 \end{cases}$, show that the derivative of $f(x)$ does not exist at the point $x = 1$.

2. $f(x) = |x^2 - 4x + 3|$ is given. Find the derivative of $f(x)$ at the point $x = 3$.

3. The graph of a function f is given below. State, with reasons, the values at which f is not differentiable.

Answers

1. compare $f'(1^-)$ and $f'(1^+)$.
2. does not exist.
3. $x = -1$, corner; $x = 4$, discontinuity; $x = 8$, corner; $x = 11$, vertical tangent.

EXERCISES

A. Tangents

1. Find the slope of the tangent line to the graph of each function at the given point.

   a. $f(x) = 5x - 1\ ;\ x = 3$

   b. $f(x) = 4 - 7x\ ;\ x = 2$

   c. $f(x) = x^2 - 1\ ;\ x = -1$

   d. $f(x) = 3x^2 - 2x - 5\ ;\ x = 0$

   e. $f(x) = x^3 - 3x + 5\ ;\ x = 1$

2. Find the equation of the tangent line to each function at the given point.

   a. $f(x) = 2x + 5$ at $(2, 9)$

   b. $f(x) = x^2 + x + 1$ at $(1, 3)$

C. Left-Hand and Right-Hand Derivatives

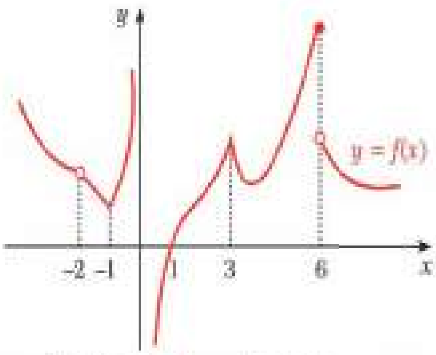
3. Let $f(x) = \begin{cases} 6x, & 0 \le x \le 8 \\ 9x - 24, & 8 < x \end{cases}$.

   Does the function have derivative at $x = 8$? Why or why not?

4. Given that $f(x) = |x - 1|$, find $f'(1)$.

D. Differentiability and Continuity

5.



The graph of $f(x)$ is given. At what numbers is $f(x)$ not differentiable? Why?

6. Let $f(x) = \begin{cases} x^2 + 7x, & x \le 1 \\ 9x - 24, & x > 1 \end{cases}$.

   Does the function have derivative at $x = 1$? Why or why not?

TECHNIQUES OF DIFFERENTATION

A. BASIC DIFFERENTIATION RULES

Up to now, we have calculated the derivatives of functions by using the definition of the derivative as the limit of a difference quotient. This method works, but it is slow even for quite simple functions. Clearly we need a simpler, quicker method. In this section, we begin to develop methods that greatly simplify the process of differentiation. From now on, we will use the notation $f'(x)$ (f prime of x) to mean the derivative of f with respect to x. Other books and mathematicians sometimes use different notation for the derivative, such as

$$\frac{d}{dx}f(x) = y' = \frac{dy}{dx} = D_x(f(x)).$$

All of these different types of notation have essentially the same meaning: the derivative of a function with respect to x. Finding this derivative is called differentiating the function with respect to x.

In stating the following rules, we assume that the functions f and g are differentiable.

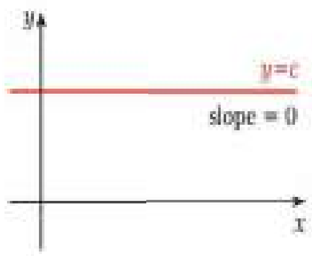
Our first rule states that the derivative of a constant function is equal to zero.

THE DERIVATIVE OF A CONSTANT FUNCTION
If c is any real number, then $c' = 0$.

We can see this by considering the graph of the constant function $f(x) = c$, which is a horizontal line. The tangent line to a straight line at any point on the line coincides with the straight line itself. So, the slope of the tangent line is zero, and therefore the derivative is zero.

We can also use the definition of the derivative to prove this result:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{c-c}{h} = \lim_{h\to 0} 0 = 0.$$



The slope of the tangent to the graph of $f(x) = c$, where c is constant, is zero.

Example 14

a. If $f(x) = 13$, then $f'(x) = (13)' = 0$.

b. If $f(x) = -\frac{1}{2}$, then $f'(x) = \left(-\frac{1}{2}\right)' = 0$.

Next we consider how to find the derivative of any power function $f(x) = x^n$.

Note that the rule applies not only to functions like $f(x) = x^3$, but also to those such as

$g(x) = \sqrt[3]{x^2}$ and $h(x) = \dfrac{1}{x^5} = x^{-5}$.

THE DERIVATIVE OF A POWER FUNCTION (POWER RULE)

If n is any real number, then $(x^n)' = nx^{n-1}$.

Example 15

a. If $f(x) = x$, then $f'(x) = x' = 1 \cdot x^{1-1} = 1$.

b. If $f(x) = x^2$, then $f'(x) = (x^2)' = 2 \cdot x^{2-1} = 2x$.

c. If $f(x) = x^3$, then $f'(x) = (x^3)' = 3 \cdot x^{3-1} = 3x^2$.

Note

To differentiate a function containing a radical expression, we first convert the radical expression into exponential form, and then differentiate the exponential form using the Power Rule.

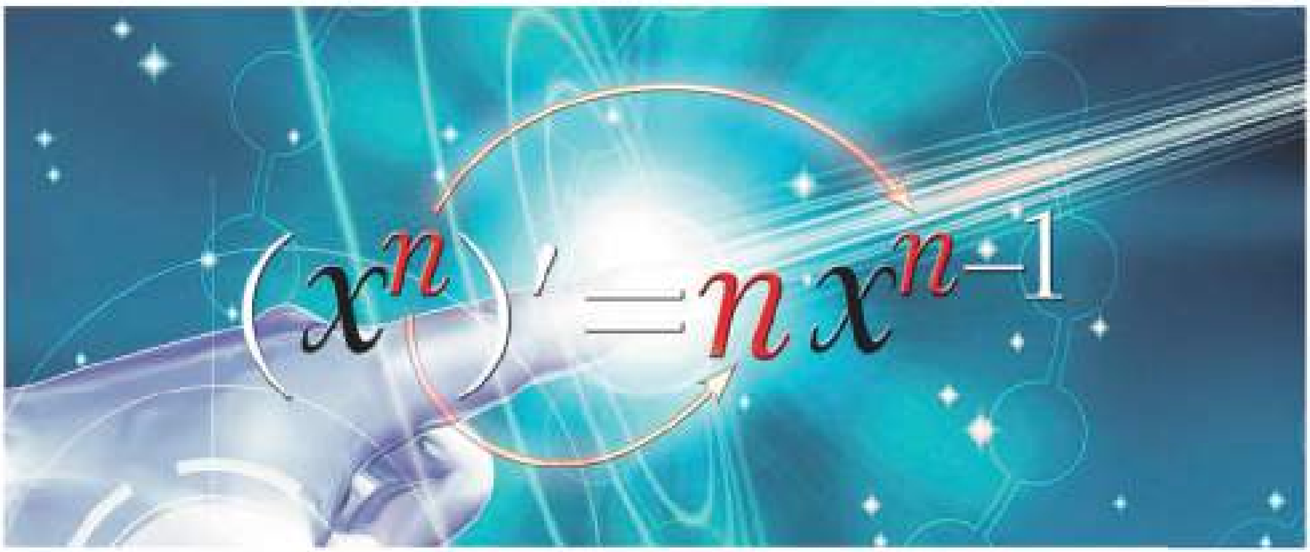
Example 16

a. If $f(x) = \sqrt[2]{x^3}$, then $f(x) = x^{3/2}$ in exponential form

$$f'(x) = (x^{3/2})' = \frac{3}{2}x^{3/2-1} = \frac{3}{2}x^{1/2}.$$

b. If $f(x) = \dfrac{1}{x}$, then $f(x) = x^{-1}$ in exponential form

$$f'(x) = (x^{-1})' = -1 \cdot x^{-1-1} = -x^{-2} = -\frac{1}{x^2}.$$



The proof of the Power Rule for the general case ($n \in \square$) is not easy to prove and will no be given here. However, we can prove the Power Rule for the case where n is a positive integer.

Proof (Power Rule) If $f(x) = x^n$, then $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} \dfrac{(x+h)^n - x^n}{h}$.

Here we need to expand $(x + h)^n$ and we use the Binomial Theorem to do so:

$$f'(x) = \lim_{h \to 0} \frac{\left[x^n + nx^{n-1}h + \dfrac{n \cdot (n-1)}{2} x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n\right] - x^n}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{n \cdot x^{n-1}h + \dfrac{n \cdot (n-1)}{2} x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h}$$

(every term includes h as a factor, so h's can be simplified)

$$f'(x) = \lim_{h \to 0} \left[nx^{n-1} + \frac{n \cdot (n-1)}{2} x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}\right] = n \cdot x^{n-1}$$

(if $h = 0$, then every term including h as a factor will be zero)

Check Yourself

Differentiate each function by using either the Constant Rule or the Power Rule.

1. $f(x) = 2$
2. $f(x) = 0.5$
3. $f(x) = -\dfrac{1}{3}$
4. $f(x) = \dfrac{\sqrt{3}}{2}$
5. $f(x) = x^3$
6. $f(x) = \sqrt[3]{x^7}$
7. $f(x) = \dfrac{1}{x^2}$
8. $f(x) = \dfrac{1}{\sqrt{x^3}}$

Answers

1. 0 2. 0 3. 0 4. 0 5. $3x^2$ 6. $\dfrac{7}{3}\sqrt[3]{x^4}$ 7. $-\dfrac{2}{x^3}$ 8. $-\dfrac{3}{2\sqrt{x^5}}$

The next rule states that the derivative of a constant multiplied by a differentiable function is equal to the constant times the derivative of the function.

THE CONSTANT MULTIPLE RULE

$$[c \cdot f(x)]' = c \cdot f'(x) \quad , \quad c \in \mathbb{R}$$

Example 17

a. If $f(x) = 3x$, then $f'(x) = (3x)' = 3 \cdot (x)' = 3 \cdot 1 = 3$.

b. If $f(x) = 3x^4$, then $f'(x) = (3x^4)' = 3(x^4)' = 3 \cdot (4x^3) = 12x^3$.

Proof (Constant Multiple Rule) If $g(x) = c \cdot f(x)$, then

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{c \cdot f(x+h) - c \cdot f(x)}{h}$$

$$g'(x) = c \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$g'(x) = c \cdot f'(x).$$

Example 18

a. If $f(x) = -\frac{2}{x^3}$, then $f'(x) = (-2x^{-3})' = -2(x^{-3})' = -2(-3x^{-4}) = 6x^{-4} = \frac{6}{x^4}$.

b. If $f(x) = 5\sqrt{x}$, then $f'(x) = (5x^{1/2})' = 5(x^{1/2})' = 5\left(\frac{1}{2}x^{-1/2}\right) = \frac{5}{2}x^{-1/2} = \frac{5}{2\sqrt{x}}$.

Next we consider the derivative of the sum or the difference of two differentiable functions. The derivative of the sum or the difference of two functions is equal to the sum or the difference of their derivatives. Note that the difference is also the sum since it deals with addition of a negative expression.

THE SUM RULE

$$[f(x) \mp g(x)]' = f'(x) \mp g'(x)$$

Note

We can generalize this rule for the sum of any finite number of differentiable functions.

$$[f(x) \mp g(x) \mp h(x) \mp \ldots]' = f'(x) \mp g'(x) \mp h'(x) \mp \ldots$$

Now, let's verify the rule for a sum of two functions.

Proof (Sum Rule) If $S(x) = f(x) + g(x)$, then

$$S'(x) = \lim_{h\to 0}\frac{S(x+h)-S(x)}{h} = \lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h}$$

$$S'(x) = \lim_{h\to 0}\frac{[f(x+h)-f(x)]+[g(x+h)-g(x)]}{h}$$

$$S'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$S'(x) = f'(x) + g'(x).$$

Example 19

a. If $f(x) = x^{-2}+7$, then $f'(x) = (x^{-2}+7)' = (x^{-2})' + (7)' = -2x^{-3}+0 = -2x^{-3}$.

b. If $g(t) = \frac{t^2}{5}+\frac{5}{t^2}$, then $g'(t) = \left(\frac{t^2}{5}+5t^{-2}\right)' = \left(\frac{t^2}{5}\right)' + (5t^{-2})' = \frac{1}{5}(t^2)' + 5(t^{-2})'$.

$$g(t)' = \frac{1}{5}(2t^{2-1}) + 5(-2t^{-2-1})$$

$$g(t)' = \frac{2}{5}t - 10t^{-3} = \frac{2t}{5} - \frac{10}{t^3}.$$

Notice that in this example, the independent variable is t instead of x. So, we differentiate the function $g(t)$ with respect to t.

By combining the Power Rule, the Constant Multiple Rule and the Sum Rule we can differentiate any polynomial. Let us look at some examples.

Example 20 Differentiate the polynomial function $f(x) = 3x^5 + 4x^4 - 7x^2 + 3x + 6$.

Solution

$$f'(x) = (3x^5 + 4x^4 - 7x^2 + 3x + 6)'$$

$$f'(x) = (3x^5)' + (4x^4)' + (-7x^2)' + (3x)' + (6)'$$

$$f'(x) = 3(x^5)' + 4(x^4)' - 7(x^2)' + 3(x)' + (6)'$$

$$f'(x) = 3\cdot 5x^4 + 4\cdot 4x^3 - 7\cdot 2x + 3\cdot 1 + 0$$

$$f'(x) = 15x^4 + 16x^3 - 14x + 3$$

Example 21 It is estimated that x months from now, the population of a certain community will be $P(x) = x^2 + 20x + 8000$.

a. At what rate will the population be changing with respect to time fifteen months from now?

b. How much will the population actually change during the sixteenth month?

Solution a. The rate of change of the population with respect to time is the derivative of the population function, i.e.

rate of change $= P'(x) = 2x + 20$.

Fifteen months from now the rate of change of the population will be:

$P'(15) = 2 \cdot 15 + 20 = 50$ people per month.

b. The actual change in the population during the sixteenth month is the difference between the population at the end of sixteen months and the population at the end of fifteen months. Therefore,

$$\text{the change in population} = P(16) - P(15) = 8576 - 8525 = 51 \text{ people.}$$

Check Yourself

1. Find the derivative of each function with respect to the variable.

a. $f(x) = \dfrac{3}{2x}$ b. $f(r) = \dfrac{4}{3}\pi r^3$ c. $f(x) = 0.2\sqrt{x}$

d. $f(x) = 3x^2 + 5x - 1$ e. $f(t) = \dfrac{4}{t^3} - \dfrac{t^2}{3} + t$ f. $f(x) = \dfrac{x^3 - 4x^2 + 3}{x}$

2. Find the derivative of $f(x) = \dfrac{x^3 - 3x^2 + 3x - 1}{x - 1}$.

3. Differentiate $f(x) = \dfrac{x^2\sqrt{x} - \sqrt{x}}{x\sqrt{x} + \sqrt{x}}$.

Answers

1. a. $-\dfrac{3}{2x^2}$ b. $4\pi r^2$ c. $\dfrac{0.1}{\sqrt{x}}$ d. $6x + 5$ e. $-\dfrac{12}{t^4} - \dfrac{2t}{3} + 1$ f. $2x - 4 - \dfrac{3}{x^2}$

2. $2x - 2$ 3. 1

B. THE PRODUCT AND THE QUOTIENT RULES

Now we learn how to differentiate a function formed by multiplication or division of functions. Based on your experience with the Constant Multiple and Sum Rules we learned in the preceding part, you may think that the derivative of the product of functions is the product of separate derivatives, but this guess is wrong. The correct formula was discovered by Leibniz and is called the Product Rule.

The Product Rule states that the derivative of the product of two functions is the derivative of the first function times the second function plus the first function times the derivative of the second function.

THE PRODUCT RULE

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$$

Be careful! The derivative of the product of two functions is not equal to the product of the derivatives:

We can easily see this by looking at a particular example.

Let $f(x) = x$ and $g(x) = x^2$. Then

$[f(x)g(x)]' \neq f'(x)g'(x)$

$$f(x)g(x) = x \cdot x^2 = x^3 \qquad f'(x) = 1 \text{ and } g'(x) = 2x$$

$$[f(x)g(x)]' = 3x^2 \qquad f'(x)g'(x) = 1 \cdot 2x = 2x$$

$$[f(x)g(x)]' \neq f'(x)g'(x).$$

Example 22 Find the derivative of the function $f(x) = x(x + 1)$.

Solution By the Product Rule,

$f'(x) = x \cdot (x+1)' + (x)' \cdot (x+1) = x \cdot 1 + 1 \cdot (x+1) = 2x + 1.$

We can check this result by using direct computation:

$f(x) = x(x + 1) = x^2 + x$ so, $f'(x) = 2x + 1$, which is the same result.

Note that preferring direct differentiation when it is easy to expand the brackets is always simpler than applying the Product Rule.

Example 23 Differentiate the function $f(x) = (2x^2 + 1)(x^2 - x)$.

Solution

$$f'(x) = (2x^2+1)' \cdot (x^2 - x) + (2x^2+1) \cdot (x^2 - x)'$$

$$f'(x) = (4x)(x^2 - 1) + (2x^2+1)(2x-1)$$

$$f'(x) = 4x^3 - 4x + 4x^3 - 2x^2 + 2x - 1$$

$$f'(x) = 8x^3 - 2x^2 - 2x - 1$$

Example 24 Differentiate the function $f(x) = (x^3 + x - 2)(2\sqrt{x} + 1)$.

Solution First, we convert the radical part into exponential form:

$$f(x) = (x^3 + x - 2)(2\sqrt{x} + 1) = (x^3 + x - 2)\cdot(2x^{1/2} + 1).$$

Now, by the Product Rule,

$$f'(x) = (x^3 + x - 2)'\cdot(2x^{1/2} + 1) + (x^3 + x - 2)\cdot(2x^{1/2} + 1)'$$

$$f'(x) = (3x^2 + 1)(2x^{1/2} + 1) + (x^3 + x - 2)\cdot x^{-1/2} = 6x^{5/2} + 3x^2 + 2x^{1/2} + 1 + x^{5/2} + x^{1/2} - 2x^{-1/2}$$

$$f'(x) = 7x^{5/2} + 3x^2 + 3x^{1/2} - 2x^{-1/2} + 1.$$

Let us look at the proof of the Product Rule.

Proof (Product Rule) If $P(x) = f(x)g(x)$, then

$$P'(x) = \lim_{h\to 0}\frac{P(x+h) - P(x)}{h} = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

By adding $-f(x+h)g(x) + f(x+h)g(x)$ (which is zero) to the numerator and factoring, we have:

$$P'(x) = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$P'(x) = \lim_{h\to 0}\frac{f(x+h)\left[g(x+h) - g(x)\right] + g(x)\left[f(x+h) - f(x)\right]}{h}$$

$$P'(x) = \lim_{h\to 0}\left(f(x+h)\left[\frac{g(x+h) - g(x)}{h}\right] + g(x)\left[\frac{f(x+h) - f(x)}{h}\right]\right)$$

$$P'(x) = \lim_{h\to 0} f(x+h)\cdot\lim_{h\to 0}\frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} g(x)\cdot\lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$P'(x) = f(x)\cdot g'(x) + g(x)\cdot f'(x) = f'(x)g(x) + f(x)g'(x).$$

Example 25 Differentiate the function $f(x) = (x^2 + 1)(3x^4 - 5x)(x^3 + 2x^2 + 4)$.

Solution In this example we have a product of three functions, but we are only able to apply the rule for the product of two functions. So, before we proceed we must imagine the function as a product of two functions as follows:

$$f(x) = (x^2 + 1)(3x^4 - 5x)(x^3 + 2x^2 + 4)$$

$$f'(x) = \underbrace{[(x^2 + 1)(3x^4 - 5x)]'}_{\text{requires product rule once more}}(x^3 + 2x^2 + 4) + (x^2 + 1)(3x^4 - 5x)(x^3 + 2x^2 + 4)'$$

$$f'(x) = [2x(3x^4 - 5x) + (x^2 + 1)(12x^3 - 5)](x^3 + 2x^2 + 4) + (x^2 + 1)(3x^4 - 5x)(3x^2 + 4x).$$

Our aim is to introduce this method and because any further simplification is time consuming, we will stop at this point.

The derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator. Or,

$$\left(\frac{\text{numerator}}{\text{denominator}}\right)' = \frac{\text{derivative of the numerator} \times \text{denominator} - \text{numerator} \times \text{derivative of the denominator}}{\text{the square of the denominator}}.$$

THE QUOTIENT RULE

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} \quad , \quad g(x) \neq 0$$

$$\left(\frac{f(x)}{g(x)}\right)' \neq \frac{f'(x)}{g'(x)}$$

The quotient rule is probably the most complicated formula you will have to learn in this text. It may help if you remember that the quotient rule resembles the Product Rule.

Also note that like in the Product Rule, the derivative of a quotient is not equal to the quotient of derivatives.

Example 26 Find the derivative of the function $f(x) = \frac{3x+1}{2x-1}$.

Solution Using the Quotient Rule:

$$f'(x) = \frac{(3x+1)'(2x-1) - (3x+1)(2x-1)'}{(2x-1)^2}$$

$$f'(x) = \frac{3 \cdot (2x-1) - (3x+1) \cdot 2}{(2x-1)^2} = \frac{6x-3-6x-2}{(2x-1)^2}$$

$$f'(x) = -\frac{5}{(2x-1)^2}.$$

Example 27 Differentiate the rational function $f(x) = \frac{x^2+x-21}{x-1}$.

Solution According to the Quotient Rule,

$$f'(x) = \frac{(2x+1) \cdot (x-1) - (x^2+x-21) \cdot 1}{(x-1)^2}$$

$$f'(x) = \frac{2x^2-x-1-x^2-x+21}{(x-1)^2} = \frac{x^2-2x+20}{(x-1)^2}$$

$$f'(x) = \frac{x^2-2x+20}{x^2-2x+1}.$$

Example 28 Differentiate the function $f(x)=\dfrac{2x^2+3x+1}{2x}$.

Solution Before trying to use the Quotient Rule let us simplify the formula of the function:

$$f(x)=\frac{2x^2+3x+1}{2x}=\frac{2x^2}{2x}+\frac{3x}{2x}+\frac{1}{2x}=x+\frac{3}{2}+\frac{1}{2}x^{-1}.$$

In this example, finding the derivative will be easier and quicker without using the Quotient Rule.

$$f'(x)=1+0-\frac{1}{2}x^{-2}=1-\frac{1}{x^2}=\frac{x^2-1}{x^2}$$

Note

We do not need to use the Quotient Rule every time we differentiate a quotient. Sometimes performing division gives us an expression which is easier to differentiate than the quotient.

Let us verify the Quotient Rule.

Proof (Quotient Rule) Let $Q(x)=\dfrac{f(x)}{g(x)}$ and $Q(x)$ be differentiable.

We can write $f(x) = Q(x)g(x)$.

If we apply the Product Rule: $f'(x) = Q'(x)g(x) + Q(x)g'(x)$

Solving this equation for $Q'(x)$, we get

$$Q'(x)=\frac{f'(x)-Q(x)g'(x)}{g(x)}=\frac{f'(x)-\dfrac{f(x)}{g(x)}\cdot g'(x)}{g(x)}$$

$$Q'(x)=\frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}.$$

Example 29 $f(x)=\sqrt{x}\cdot g(x)$, where $g(4)=2$ and $g'(4)=3$. Find $f'(4)$.

Solution $f'(x)=(\sqrt{x}\cdot g(x))'=(\sqrt{x})'\cdot g(x)+\sqrt{x}\cdot g'(x)$

$$f'(x)=\frac{g(x)}{2\sqrt{x}}+\sqrt{x}\cdot g'(x)$$

So $f'(4)=\dfrac{g(4)}{2\sqrt{4}}+\sqrt{4}\cdot g'(4)=\dfrac{2}{2\cdot 2}+2\cdot 3=\dfrac{13}{2}.$

Check Yourself

1. Find the derivative of each function using the Product or the Quotient Rule.

a. $f(x) = 2x(x^2 + x + 1)$

b. $f(x) = (x^3 - 1)(x^2 - 2)$

c. $f(x) = \left(\frac{1}{x^2} + x\right)\left(\frac{1}{x} + 1\right)$

d. $f(x) = (\sqrt{x} + 1)\left(x^2 + \frac{1}{\sqrt{x}}\right)$

e. $f(x) = \frac{2x + 4}{3x - 1}$

f. $f(x) = \frac{\sqrt{x} - 1}{\sqrt{x} + 1}$

g. $f(x) = \frac{x^2 - x + 10}{x + 1}$

h. $f(x) = \frac{x^3 + 3x^2 - 5x + 6}{2x}$

2. If $f(x)$ is a differentiable function, find an expression for the derivative of each function.

a. $y = x^2 f(x)$ b. $y = \frac{f(x)}{x^2}$ c. $y = \frac{x^2}{f(x)}$ d. $y = \frac{1 + xf(x)}{x}$

3. Suppose that f and g are two functions such that $f(5) = 1$, $f'(5) = 6$, $g(5) = -3$ and $g'(5) = 2$. Find each value.

a. $(fg)'(5)$ b. $\left(\frac{f}{g}\right)'(5)$ c. $\left(\frac{g}{f}\right)'(5)$

Answers

1. a. $6x^2 + 4x + 2$ b. $5x^4 - 6x^2 - 2x$ c. $1 - \frac{2}{x^3} - \frac{3}{x^4}$ d. $\frac{5}{2}x\sqrt{x} + 2x - \frac{1}{2x\sqrt{x}}$

e. $-\frac{14}{(3x - 1)^2}$ f. $\frac{1}{\sqrt{x}(\sqrt{x} + 1)^2}$ g. $\frac{x^2 + 2x - 11}{(x + 1)^2}$ h. $x - \frac{3}{x^2} + \frac{3}{2}$

2. a. $2xf(x) + x^2 f'(x)$ b. $\frac{f'(x) \cdot x^2 - 2xf(x)}{x^4}$ c. $\frac{2xf(x) - f'(x) \cdot x^2}{(f(x))^2}$ d. $\frac{x^2 f'(x) - 1}{x^2}$

3. a. -16 b. $-\frac{20}{9}$ c. 20

C. THE CHAIN RULE

We have learned how to find the derivatives of expressions that involve the sum, difference, product or quotient of different powers of x. Now consider the function given below.

$$h(x) = (x^2 + x - 1)^{50}$$

In order to differentiate $h(x)$ using the rules we know, we need to expand $h(x)$, then find the derivative of each separate term. This method is, however, tedious!

Consider also the function $m(x) = \sqrt{x^2 + x - 1}$. This function is also difficult to differentiate using the rules we have learned. For each of the two functions $h(x)$ and $m(x)$, the differentiation formula we learned in the previous sections cannot be applied easily to calculate the derivatives $h'(x)$ and $m'(x)$.

We know that both h and m are composite functions because both are built up from simpler functions. For example, the function $h(x) = (x^2 + x - 1)^{50}$ is built up from the two simpler functions $f(x) = x^{50}$ and $g(x) = x^2 + x - 1$ like this:

$$h(x) = f(g(x)) = [g(x)]^{50} = (x^2 + x - 1)^{50}$$

Here we know how to differentiate both f and g, so it would be useful to have a rule that tells us how to find the derivative of $h = f(g(x))$ in terms of the derivatives of f and g.

THE CHAIN RULE

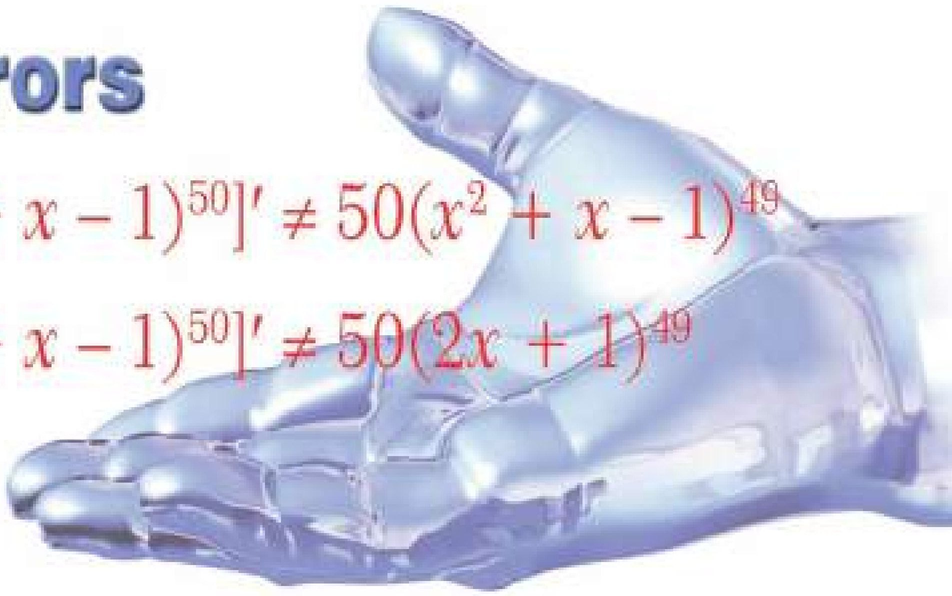
$$[f(g(x))]' = f'(g(x)) \cdot g'(x)$$

- f: outer function
- $g(x)$: argument of the outer function
- $f'(g(x))$: derivative of the outer function
- $g'(x)$: derivative of the argument of the outer function

For example, if $h(x) = f(g(x)) = (x^2 + x - 1)^{50}$, then

$h'(x) = f'(g(x)) \cdot g'(x) = 50(x^2 + x - 1)^{49} \cdot (2x + 1)$.

Common Errors



1. $[(x^2 + x - 1)^{50}]' \neq 50(x^2 + x - 1)^{49}$
2. $[(x^2 + x - 1)^{50}]' \neq 50(2x + 1)^{49}$

Example 30 Differentiate the function $f(x) = (3x^2 + 5x)^{2005}$.

Solution By the Chain Rule, $f'(x) = 2005(3x^2+5x)^{2004} \cdot (3x^2+5x)' = 2005(3x^2+5x)^{2004} \cdot (6x+5)$.

Example 31 Suppose $m(x) = f(g(x))$ and $g(1) = 5$, $g'(1) = 2$, $f(5) = 3$ and $f'(5) = 4$ are given. Find $m'(1)$.

Solution By the Chain Rule, $m'(x) = f'(g(x)) \cdot g'(x)$. So $m'(1) = f'(g(1)) \cdot g'(1) = f'(5) \cdot 2 = 4 \cdot 2 = 8$.

Note

The Chain Rule can be generalized for the composition of more than two functions as follows:

$$[f_1(f_2(f_3(...f_n(x)...)))]' = [f_1'(f_2(f_3(...f_n(x)...)))] \cdot (f_2'(f_3(...f_n(x)...))) \cdot (f_3'(...f_n(x)...)) \cdot ... \cdot f_n'(x)$$

Using the Chain Rule we can generalize the Power Rule as follows:

GENERAL POWER RULE

$$[(f(x))^n]' = n(f(x))^{n-1} \cdot f'(x)$$

By using this rule we can more easily differentiate the functions that can be written as the power of any other functions.

Example 32 Differentiate the function $m(x) = \sqrt{x^2 + x - 1}$.

Solution We can rewrite the function as $m(x) = (x^2 + x - 1)^{\frac{1}{2}}$ and apply the General Power Rule:

$$m'(x) = \frac{1}{2}(x^2 + x - 1)^{-\frac{1}{2}} \cdot (x^2 + x - 1)'$$

$$m'(x) = \frac{1}{2}(x^2 + x - 1)^{-\frac{1}{2}} \cdot (2x + 1)$$

$$m'(x) = \frac{2x+1}{2\sqrt{x^2 + x - 1}}$$

Example 33 Differentiate the function $f(x) = \dfrac{1}{x^2 + 3x}$.

Solution

$$f'(x) = [(x^2 + 3x)^{-1}]' = -1(x^2 + 3x)^{-2} \cdot (x^2 + 3x)'$$

$$f'(x) = -1(x^2 + 3x)^{-2} \cdot (2x + 3)$$

$$f'(x) = -\frac{2x+3}{(x^2 + 3x)^2}$$

Example 34 Differentiate the function $f(x) = (2x^3 + x^2 - 15)^{-\frac{1}{3}}$.

Solution

$$f'(x) = -\frac{1}{3}(2x^3 + x^2 - 15)^{-\frac{4}{3}} \cdot (2x^3 + x^2 - 15)'$$

$$f'(x) = -\frac{1}{3} \cdot \frac{1}{\sqrt[3]{(2x^3 + x^2 - 15)^4}} \cdot (6x^2 + 2x)$$

$$f'(x) = -\frac{6x^2 + 2x}{3\sqrt[3]{(2x^3 + x^2 - 15)^4}}$$

Example 35 Differentiate the function $f(x) = ((x+1)^{-\frac{2}{3}} + 5x)^{-3}$.

Solution

$$f'(x) = -3((x+1)^{-\frac{2}{3}} + 5x)^{-4} \cdot ((x+1)^{-\frac{2}{3}} + 5x)'$$

$$f'(x) = -3((x+1)^{-\frac{2}{3}} + 5x)^{-4} \cdot (-\frac{2}{3}(x+1)^{-\frac{5}{3}} \cdot (x+1)' + 5)$$

$$f'(x) = -3((x+1)^{-\frac{2}{3}} + 5x)^{-4} \cdot (5 - \frac{2}{3}(x+1)^{-\frac{5}{3}})$$

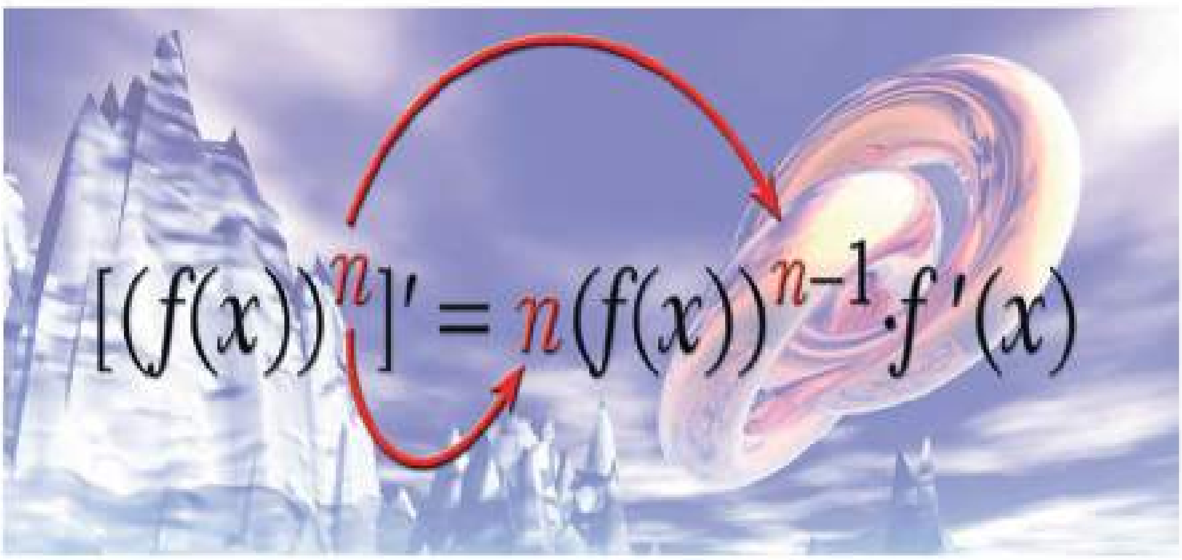
Example 36 Differentiate the function $f(x) = (2x-3)^5 \cdot \sqrt{x^2 - 2x}$.

Solution The function is the product of two expressions, so we can use the Product Rule:

$$f'(x) = ((2x-3)^5)' \cdot \sqrt{x^2 - 2x} + (2x-3)^5 \cdot (\sqrt{x^2 - 2x})'$$

$$f'(x) = 5 \cdot (2x-3)^4 \cdot 2 \cdot \sqrt{x^2 - 2x} + (2x-3)^5 \cdot \frac{1}{2} \cdot (x^2 - 2x)^{-\frac{1}{2}} \cdot (2x-2)$$

$$f'(x) = 10(2x-3)^4 \sqrt{x^2 - 2x} + \frac{(2x-3)^5 \cdot (2x-2)}{2\sqrt{x^2 - 2x}}$$



Example 37 Differentiate the function $g(t)=\left(\frac{2t+1}{t-3}\right)^7$.

Solution

$$g'(t)=7\left(\frac{2t+1}{t-3}\right)^6\cdot\left(\frac{2t+1}{t-3}\right)'$$ *(by the Power Rule)*

$$g'(t)=7\left(\frac{2t+1}{t-3}\right)^6\cdot\frac{2\cdot(t-3)-1\cdot(2t+1)}{(t-3)^2}$$ *(by the Quotient Rule)*

$$g'(t)=7\left(\frac{2t+1}{t-3}\right)^6\cdot\frac{2t-6-2t-1}{(t-3)^2}$$ *(simplify)*

$$g'(t)=\frac{-49(2t+1)^6}{(t-3)^8}.$$

Notation

Remember that if $y=f(x)$, then we can denote its derivative by y' or $\frac{dy}{dx}$.

If $y=f(g(x))$ such that $y=f(u)$ and $u=g(x)$, then we can denote the derivative of $f(g(x))$ by $y'=f'(g(x))\cdot g'(x)$ or $y'=f'(u)\cdot u'(x)$ or $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$.

The notation $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$ is called Leibniz notation for the Chain Rule.

Example 38 Given that $y=u^2-1$ and $u=3x^2+1$, find $\frac{dy}{dx}$ by using the Chain Rule.

Solution By the Chain Rule,

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

$$\frac{dy}{dx}=\frac{d}{du}(u^2-1)\cdot\frac{d}{dx}(3x+1)$$ *(find the derivative of the first function with respect to u and the second function with respect to x)*

$$\frac{dy}{dx}=(2u-1)\cdot3$$

$$\frac{dy}{dx}=(2\cdot(3x+1)-1)\cdot3$$

$$\frac{dy}{dx}=18x+3.$$

Check Yourself

1. Find the derivative of $f(x) = (2x + 1)^3$.
2. Differentiate $y = (x^3 - 1)^{100}$.
3. Find $f'(x)$ given $f(x) = \dfrac{1}{\sqrt[3]{x^2 + x + 1}}$.
4. Find the derivative of $g(x) = \sqrt[4]{\dfrac{x^3 - 1}{x^3 + 1}}$.
5. $y = \dfrac{1}{u}$ and $u = 3x - 1$ are given. Find $\dfrac{dy}{dx}$.

Answers

1. $6(2x + 1)^2$
2. $300x^2(x^3 - 1)^{99}$
3. $\dfrac{2x+1}{3\sqrt[3]{(x^2 + x + 1)^4}}$
4. $\dfrac{1}{4}\left(\dfrac{x^3+1}{x^3-1}\right)^{\frac{3}{4}} \dfrac{6x^2}{(x^3+1)^2}$
5. $-\dfrac{3}{(3x-1)^2}$

EXERCISES

A. Basic Differentiation Rules

1. Find the derivative of each function by using the rules of differentiation.

a. $f(x) = \sqrt{2}$

b. $f(x) = -\dfrac{1}{151}$

c. $f(x) = e^{e}$

d. $f(x) = \dfrac{1}{12}x^{8}$

e. $f(x) = 2x^{0.8}$

f. $f(x) = \dfrac{5}{4}x^{4/5}$

g. $f(x) = \dfrac{2}{\sqrt[6]{4^{11}}}$

h. $f(x) = 0.3x^{0.7}$

i. $f(x) = 7x^{-12}$

j. $f(x) = 5x^2 - 3x + 7$

k. $f(x) = \dfrac{x^3 + 2x^2 + x - 1}{x}$

l. $f(x) = \dfrac{4}{t^4} - \dfrac{3}{t^3} + \dfrac{2}{t}$

m. $f(x) = \sqrt{x} + \sqrt[3]{x} + \sqrt[5]{x}$

n. $f(x) = \sqrt{x} + \dfrac{2}{\sqrt{x}} + \dfrac{1}{x}$

o. $f(x) = 1 - \dfrac{1}{x} + \dfrac{3}{\sqrt{x}}$

EXERCISES

B. The Product and The Quotient Rules

2. Find the derivative of each function by using the Product or the Quotient Rule.

a. $f(x) = 5x(x^2 - 1)$

b. $f(x) = (2x + 3)(3x - 4)$

c. $f(x) = 10(3x + 1)(1 - 5x)$

d. $f(x) = (x^3 - 1)(x + 1)$

e. $f(x) = (x^3 - x^2 + x - 1)(x^2 + 2)$

f. $f(x) = (1 + \sqrt{t})(2t^2 - 3)$

g. $f(x) = \dfrac{3}{2x+4}$

h. $f(x) = \dfrac{x-1}{2x+1}$

i. $f(x) = \dfrac{1-2x}{1+3x}$

j. $f(x) = \dfrac{\sqrt{x}}{x^2+1}$

k. $f(x) = \dfrac{x^2+2}{x^2+x+1}$

l. $f(x) = \dfrac{x+\sqrt{3x}}{3x-1}$

3. Given that $f(1) = 2$, $f'(1) = -1$, $g(1) = -2$ and $g'(1) = 3$, find the value of $h'(1)$.

a. $h(x) = f(x) \cdot g(x)$

b. $h(x) = (x^2 + 1) \cdot g(x)$

c. $h(x) = \dfrac{xf(x)}{x+g(x)}$

d. $h(x) = \dfrac{f(x) \cdot g(x)}{f(x) - g(x)}$

4. Differentiate the function $f(x)=\dfrac{x-3x\sqrt{x}}{\sqrt{x}}$ by simplifying and by the Quotient Rule. Show that both of your answers are equivalent. Which method do you prefer? Why?

5. $f(3) = 4$, $g(3) = 2$, $f'(3) = -6$ and $g'(3) = 5$ are given. Find the value of the following expressions.

a. $(f + g)'(3)$ b. $(fg)'(3)$

c. $\left(\dfrac{f}{g}\right)'(3)$ d. $\left(\dfrac{f}{f-g}\right)(3)$

C. The Chain Rule

6. Find the derivative of each function.

a. $f(x) = (3x - 1)^2$

b. $f(x) = (x^2 + 2)^5$

c. $f(x) = (x^5 - 3x^2 + 6)^7$

d. $f(x) = (x - 2)^{-3}$

e. $f(x)=\dfrac{2}{(5x^2+3x-1)^2}$

f. $f(x)=\dfrac{1}{\sqrt{4x^2+1}}$

g. $f(x)=(\sqrt{x+1}+\sqrt{x})^3$

oh. $f(x) = (x - 1)^5 \cdot (3x + 1)^{1/3}$

oi. $f(x)=\dfrac{(1-3x)^7}{(2x+1)^4}$

oj. $f(x)=(\dfrac{3x-9}{2x+4})^3$

ok. $f(x)=\sqrt{\dfrac{2x-1}{3x+1}}$

ol. $f(x) = 3x + [2x^2 + (x^3 + 1)^2]^{3/4}$

7. $h(x) = g(f(x))$ and $f(2) = 3$, $f'(2) = -3$, $g(3) = 5$ and $g'(3) = 4$ are given. Find $h'(2)$.

8. By using the Chain Rule, find $\dfrac{dy}{dx}$ for each function.

a. $y = u^2 - 1$, $u = 2x + 1$

b. $y = u^2 + 2u + 2$, $u = x - 1$

c. $y=\dfrac{1}{u-1}$, $u = x^3$

d. $y=\sqrt{u}+\dfrac{1}{\sqrt{u}}$, $u = x^2 - x$

D- DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

Let us begin by looking at the derivatives of the sine and cosine functions.

DERIVATIVES OF SINE AND COSINE

$$(\sin x)' = \cos x$$

$$(\cos x)' = -\sin x$$

Example 39 Find the derivative of the function $f(x) = (\sin x + \cos x)^2$.

Solution

$f'(x) = 2(\sin x + \cos x)(\sin x + \cos x)'$ *(by the General Power Rule)*

$f'(x) = 2(\sin x + \cos x)(\cos x - \sin x)$ *(by the sum, derivative of the sine and cosine)*

$f'(x) = 2(\cos^2 x - \sin^2 x)$ *(simplify)*

$f'(x) = 2\cos 2x$ *(by the trigonometric identity)*

$\cos 2x = \cos^2 x - \sin^2 x$

Now let us derive the formula for the derivative of the function $f(x) = \sin x$.

Proof (Derivative of Sine Function)

By the definition of the derivative, we have

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h} = \lim_{h\to 0}\left(\frac{\sin x\cos h - \sin x}{h} + \frac{\cos x\sin h}{h}\right)$$

$$f'(x) = \lim_{h\to 0}\left[\sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right)\right] = \lim_{h\to 0}\sin x \cdot \lim_{h\to 0}\frac{\cos h - 1}{h} + \lim_{h\to 0}\cos x \cdot \lim_{h\to 0}\frac{\sin h}{h}$$

$$f'(x) = \sin x \cdot 0 + \cos x \cdot 1 = \cos x.$$

$\lim_{h\to 0}\frac{\sin h}{h} = 1$

$\lim_{h\to 0}\frac{\cos h - 1}{h} = 0$

Example 40 Find the derivative of the function $f(x) = x \cdot \sin x$.

Solution By the Product Rule

$$f'(x) = (x \cdot \sin x)' = (x)' \cdot \sin x + x \cdot (\sin x)' = \sin x + x\cos x$$

CHAIN RULE FOR SINE AND COSINE

$$(\sin f(x))' = \cos f(x) \cdot f'(x)$$
$$(\cos f(x))' = -\sin f(x) \cdot f'(x)$$

Example 41 Find the derivative of the function $f(x) = \cos(x^3 - x)$.

Solution $f'(x) = (\cos(x^3 - x))' = -\sin(x^3 - x) \cdot (x^3 - x)' = -\sin(x^3 - x) \cdot (3x^2 - 1)$

Example 42 Find the derivative of the function $f(x) = \sin^5 x^2$.

Solution In this example we have the composition of three functions.

$f(x) = \sin^5 x^2 = (\sin(x^2))^5$

We apply the Chain Rule beginning from the outermost function:

$f'(x) = ((\sin(x^2))^5)' = 5(\sin(x^2))^4 \cdot (\sin(x^2))'$

$f'(x) = 5(\sin(x^2))^4 \cdot \cos(x^2) \cdot (x^2)'$

$f'(x) = 5(\sin(x^2))^4 \cdot \cos(x^2) \cdot 2x$

$f'(x) = 10x\sin(x^2)\cos(x^2)$

$\sin 2x = 2\sin x \cos x$

Chek Yourself

Find the derivative of each function

1. $f(x) = -3\sin x$
2. $f(x) = x\cos x$
3. $f(x) = \frac{\cos}{1+\cos x}$
4. $f(x) = \cos 2-(x^2+x-1)$

Answers

1. 1-3cos x
2. $\cos x - x\sin x$
3. $\frac{1}{1+\cos x}$
4. $\sin(2x^2+2r-2).(2x+1)$

DERIVATIVES OF OTHER TRIGONOMETRIC FUNCTIONS	
$(\tan x)' = \sec^2 x = 1 + \tan^2 x$	$(\tan f(x))' = \sec^2 f(x) \cdot f'(x)$
$(\cot x)' = -\csc^2 x = -(1 + \cot^2 x)$	$(\cot f(x))' = -\csc^2 f(x) \cdot f'(x)$
$(\sec x)' = \sec x \cdot \tan x = \frac{\sin x}{\cos^2 x}$	$(\sec f(x))' = \sec f(x) \cdot \tan f(x) \cdot f'(x)$
$(\csc x)' = -\csc x \cdot \cot x = -\frac{\cos x}{\sin^2 x}$	$(\csc f(x))' = -\csc f(x) \cdot \cot f(x) \cdot f'(x)$

Example 43 Find the derivative of the function $f(x) = \tan(x^2 - 3x + 1)$.

Solution $f'(x) = \sec^2(x^2 - 3x + 1) \cdot (x^2 - 3x + 1)' = \sec^2(x^2 - 3x + 1) \cdot (2x - 3)$

or $= (1 + \tan^2(x^2 - 3x + 1))(2x - 3)$.

Example 44 Find the derivative of the function $f(x)=\dfrac{\sec x}{1+\tan x}$.

Solution By the Quotient Rule,

$$f'(x)=\frac{(\sec x)'\cdot(1+\tan x)-\sec x\cdot(1+\tan x)'}{(1+\tan x)^2} \quad \textit{(by the Quotient Rule)}$$

$$f'(x)=\frac{\sec x\tan x\cdot(1+\tan x)-\sec x\cdot\sec^2 x}{(1+\tan x)^2} \quad \textit{(differentiate)}$$

$$f'(x)=\frac{\sec x(\tan x+\tan^2 x-\sec^2 x)}{(1+\tan x)^2} \quad \textit{(factorize)}$$

$$f'(x)=\frac{\sec x(\tan x-1)}{(1+\tan x)^2} \quad \textit{(simplify using } \tan^2 x+1=\sec^2 x\textit{)}$$

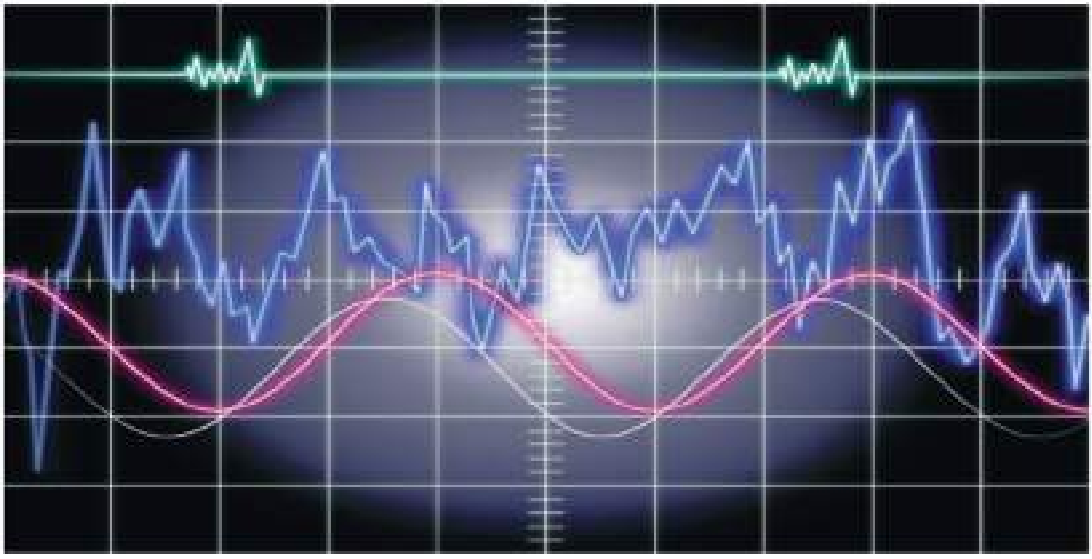
Check Yourself

Find the derivative of each function.

1. $f(x)=\dfrac{\tan x}{x}$
2. $f(x) = 4\sec x - \cot x$
3. $f(x) = \cot(x^2 - x + 1)$

Answers

1. $\dfrac{x\sec^2 x-\tan x}{x^2}$
2. $4\sec x\tan x + \csc^2 x$
3. $\csc^2(x^2 - x + 1)\cdot(1 - 2x)$



E. IMPLICIT DIFFERENTIATION

Up to now we have worked with the functions expressed in the form $y = f(x)$. In this form, the variable y is expressed easily in terms of the variable x. A function in this form is said to be in the **explicit form**. However, some functions cannot be written in explicit form. Consider the following equation:

$$y^5 + y + x = 0$$

If we are given a value of x, we can calculate y in this equation. However, we cannot write the equation in the form $y = f(x)$. We say that x determines y **implicitly**, and that y is an **implicit function** of x. Look at the same more implicit functions:

$$x^3 + 2xy^2 - 3y^4 = 7$$
$$y - 2y^3 = x$$
$$x^2 - y^3 + 4y = 0$$

How can we differentiate an implicit function? Recall the Chain Rule for differentiation. In an implicit function, y is still a function of x, even if we cannot write this explicitly. So, we can use the Chain Rule to differentiate terms containing y as functions of x. For example, if we are differentiating in terms of x,

$$(y^4)' = [(f(x))^4]' = 4(f(x))^3 f'(x) = 4y^3 y' \text{ or } (y^4)' = y^3 \frac{dy}{dx},$$
$$(7y)' = 7y' \text{ or } (7y)' = 7\frac{dy}{dx}.$$

The procedure we use for differentiating implicit functions is called **implicit differentiation**. Let us summarize the important steps involved in implicit differentiation.

IMPLICIT DIFFERENTIATION

1. Differentiate both sides of the equation with respect to x. Remember that y is really a function of x and use the Chain Rule when differentiating terms containing y.
2. Solve the resulting equation for y' or $\frac{dy}{dx}$ in terms of x and y.

Example 45 Find y' given the equation $y^5 + y + x = 0$.

Solution

$(y^5 + y + x)' = (0)'$ *(differentiate both sides)*

$(y^5)' + (y)' + (x)' = 0$ *(by the Sum Rule)*

$5y^4y' + y' + 1 = 0$ *(by the Chain Rule)*

$y'(5y^4 + 1) = -1$ *(factorize)*

$y' = -\dfrac{1}{5y^4 + 1}$ *(isolate y')*

Example 46 Find $\dfrac{dy}{dx}$ given the equation $y^3 - y^2x + x^2 - 1 = 0$.

Solution

$(y^3 - y^2x + x^2 - 1)' = (0)'$ *(differentiate both sides)*

$(y^3)' - (y^2x)' + (x^2)' - (1)' = 0$ *(by the Sum Rule)*

$3y^2\dfrac{dy}{dx} - (2y\dfrac{dy}{dx}x + y^2) + 2x - 0 = 0$ *(by the Chain Rule and the Product Rule)*

$\dfrac{dy}{dx}(3y^2 - 2yx) = y^2 - 2x$ *(factorize)*

$\dfrac{dy}{dx} = \dfrac{y^2 - 2x}{3y^2 - 2yx}$ *(isolate $\dfrac{dy}{dx}$)*

Example 47 The equation $x^2 + y^2 = 4$ is given.

a. Find $\dfrac{dy}{dx}$ by implicit differentiation.

b. Find the slope of the tangent line to the curve at the point $(\sqrt{3}, 1)$.

c. Find the equation of the tangent line at this point.

Solution a. Differentiating both sides of the equation with respect to x, we obtain

$$(x^2+y^2)'=(4)'$$

$$(x^2)'+(y^2)'=0$$

$$2x+2y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{x}{y} \quad (y\neq 0).$$

$\left.\frac{dy}{dx}\right|_{(a,\,b)}$ is used for slope of the curve at the point (a, b).

b. The slope of the tangent line to the curve at the point $(\sqrt{3}, 1)$ is given by

$$m=\left.\frac{dy}{dx}\right|_{(\sqrt{3},\,1)}=-\left.\frac{x}{y}\right|_{(\sqrt{3},\,1)}=-\frac{\sqrt{3}}{1}=-\sqrt{3}.$$

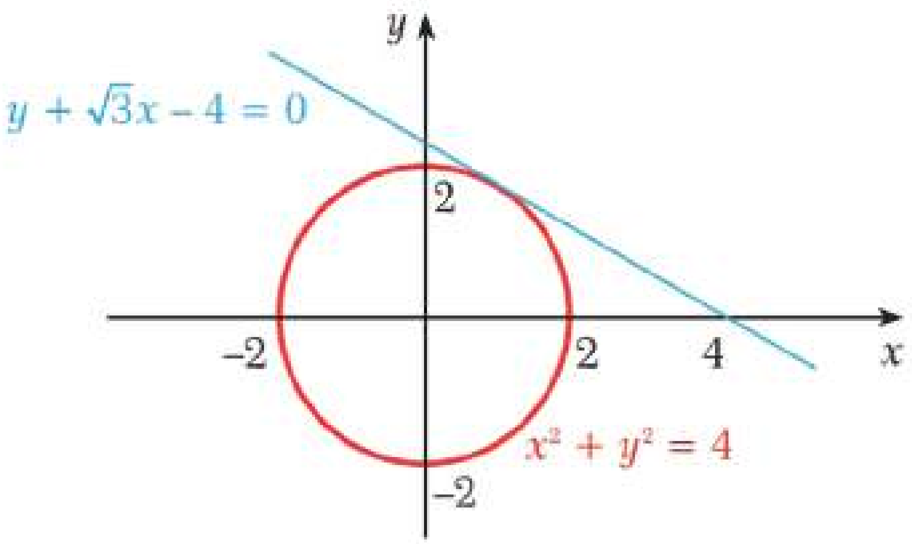
c. We can find the equation of the tangent line by using the point-slope form of the equation of a line. The slope is $m = -\sqrt{3}$ and the point is $(\sqrt{3}, 1)$. Thus,

$$y-y_1=m(x-x_1)$$

$$y-1=-\sqrt{3}(x-\sqrt{3})$$

$$\sqrt{3}x+y-4=0.$$

A sketch of this tangent line is given on the right. The line $x+\sqrt{3}y-4=0$ is tangent to the graph of the equation $x^2+y^2=4$ at the point $(\sqrt{3}, 1)$.



Example 48 Find the derivative with respect to x of the implicit function $\sqrt{x^2+y^2}+x^2=2$.

Solution Differentiating both sides of the given equation with respect to x, we obtain

$$\frac{d}{dx}(x^2+y^2)^{1/2}+\frac{d}{dx}(x^2)=\frac{d}{dx}(2)$$

$$\frac{1}{2}(x^2+y^2)^{1/2}\frac{d}{dx}(x^2+y^2)+2x=0$$

$$\frac{1}{2}(x^2+y^2)^{-1/2}(2x+2y\frac{dy}{dx})+2x=0$$

$$2x+2y\frac{dy}{dx}=-4x(x^2+y^2)^{1/2}$$

$$2y\frac{dy}{dx}=-4x(x^2+y^2)^{1/2}-2x$$

$$\frac{dy}{dx}=\frac{-2x\sqrt{x^2+y^2}-x}{y}.$$

Check Yourself

1. Find $\frac{dy}{dx}$ by implicit differentiation.

   a. $x^3+x^2y+y^2=5$ b. $x^2y+xy^2=3x$

   c. $e^xe^y=1$

2. Find the equation of the tangent line to each curve at the given point.

   a. $x^2y^3-y^2+xy-1=0$; $(1, 1)$ b. $\frac{x^2}{16}-\frac{y^2}{9}=1$; $(-5, \frac{9}{4})$

   c. $x^{\frac{2}{3}}+y^{\frac{2}{3}}=4$; $(1, 3\sqrt{3})$

Answers

1. a. $\frac{-3x^2-2xy}{x^2+2y}$ b. $\frac{3-2xy-y^2}{x^2+2xy}$

2. a. $y=-\frac{3}{2}x+\frac{5}{2}$ b. $y=-\frac{5}{4}x-4$ c. $y=-\sqrt{3}x+4\sqrt{3}$

EXERCISES

D. Derivatives of Trigonometric Functions

5. Differentiate the functions.

a. $f(x) = \sin(3x - 5)$

b. $f(x) = \cos(x^2 - 1)$

c. $f(x) = \sin x - \cos x$

d. $f(x) = 2\tan x + \sec x$

e. $f(x) = \sin x \cdot \tan x$

f. $f(x) = 2x\tan x - x\cos x$

g. $f(x) = \cos^2(2x^3 - 3x)$

h. $f(x) = (\frac{1-\cos x}{1+\cos x})^{10}$

i. $f(x) = \frac{\cot x}{1+\sec x}$

j. $f(x) = (1 + \sec x)\cdot(1 - \cos x)$

k. $f(x) = \tan\sqrt{x^2 - x - 1}$

l. $f(x) = \frac{\cot(x^3 - 1)}{1 - \sec^2(x^3 - 1)}$

m. $f(x) = [x^2\sin(x - 1)]^5$

n. $f(x) = \sec^2(\frac{x^3}{x^2 - 1})$

1. Find the equation of the tangent line to the curve at the given point.

a. $y = x\cos x$; $x = \pi$

2. For what values of x does the graph of $f(x) = x + 2\sin x$ have a horizontal tangent line?

E. Implicit Differentiation

3. Find $\frac{dy}{dx}$ for each equation below.

a. $5x - 4y = 3$ b. $xy - y - 1 = 0$

c. $x^3 + x^2 - xy = 1$ d. $\frac{y}{x} - 3x^2 = 5$

e. $2x^3 + 3y^2 = 12$ f. $x^2 + 5xy + y^3 = 11$

g. $x^2y^3 - xy = 8$ h. $\sqrt{xy} - 3x - y^3 = 0$

4. Find the equation of the tangent line to the given curve at the indicated point.

a. $4x^2 + 2y^2 = 12$; $(1, -2)$

b. $2x^2 + xy = 3y^2$; $(-1, -1)$

Chapter 7

SPACE GEOMETRY

1. INTRODUCTION TO SPACE GEOMETRY

AXIOMS OF SPACE GEOMETRY

In your previous studies, you studied plane geometry. Plane geometry is concerned with the definitions and properties of the figures in the plane. However we live in a three-dimensional world. Therefore we need to extend our study of geometry to include figures which have three dimensions, that is, figures with height or thickness as well as length and width.

Definition

line

A **line** is a one-dimensional figure. It has length, but no width or height. Two lines in the same plane can be coincident, intersecting or parallel. In plane geometry, the intersection of two non-coincident lines is a single point. Perpendicular lines intersect at 90 degrees.

Definition

plane

A **plane** is two-dimensional. It has length and width but no height, and extends infinitely on all sides. Two planes can be either parallel (with no common points) or intersecting. The intersection of two planes is a line.

Definition

space

Space is the set of all points. Space contains an infinite number of planes.

Definition

space geometry

The geometry of three-dimensional shapes is called **three-dimensional geometry** or **space geometry**.

In this chapter we will study lines and planes in space. We will state axioms, definitions and theorems about lines and planes. In the proofs of theorems, sometimes we will use theorems from plane geometry. We will use plane geometry theorems without proving them.

Definition

collinear, non-collinear, coplanar, non-coplanar

Points which are on the same line are called **collinear points.** Points which are not on the same line are called **non-collinear points**. In the same way, points or lines lying in the same plane are called **coplanar points** or **coplanar lines**. If they do not lie in the same plane they are **non-coplanar points** or **non-coplanar lines**.

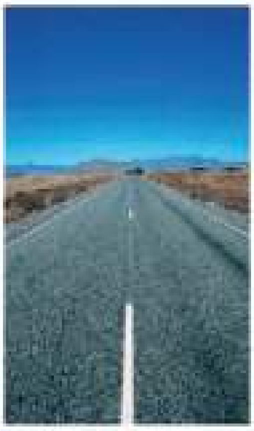
In our study of space geometry we will use the axioms of plane geometry and the axioms of space geometry. The axioms of space geometry are as follows:

Axioms of Space Geometry

1. Two different points in space determine a line.
2. Three non-collinear points in space determine a plane.
3. In space, there are at least two points on a line and there is at least one point outside this line.
4. A plane which has two points in common with a line contains this line.
5. If two planes have a common point then there is a common line passing through this point (i.e. the intersection of two intersecting planes is a line).
6. In space, outside a plane there is at least one point.
7. A plane divides a space into two semispaces.

We can use these axioms to prove theorems and corollaries about space geometry.

1. Relative Position of Two Lines in Space



Two lines in space can have different positions relative to each other. These can be described as follows:

a. Infinitely Many Intersections

The lines can be coincident. If two lines are coincident then they have infinitely many intersections. If two lines have two common points then they are coincident.

b. One Common Point

The lines can intersect. From plane geometry we know that if two non-coincident lines intersect each other then their intersection is a unique point. We have seen that two intersecting lines determine a plane. So intersecting lines always have a common plane.

c. No Common Point

If two lines have no common points then there are two possible cases:

i. **Parallel lines:** Parallel lines are defined as coplanar lines having no points in common. From plane geometry we know that in a plane, through a point not on a line we can draw one and only one line parallel to the given line. Similarly we can say that in space, through a point not on a line, a line parallel to the given line can be drawn and this line is unique.

ii. **Skew lines:** Two lines are called **skew lines** if there is no plane which contains both lines. In other words, skew lines cannot be coplanar. By extension, skew lines have no common point.

2. Relative Position of a Plane and a Line

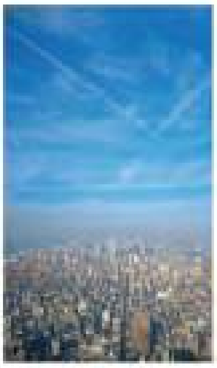
Rule

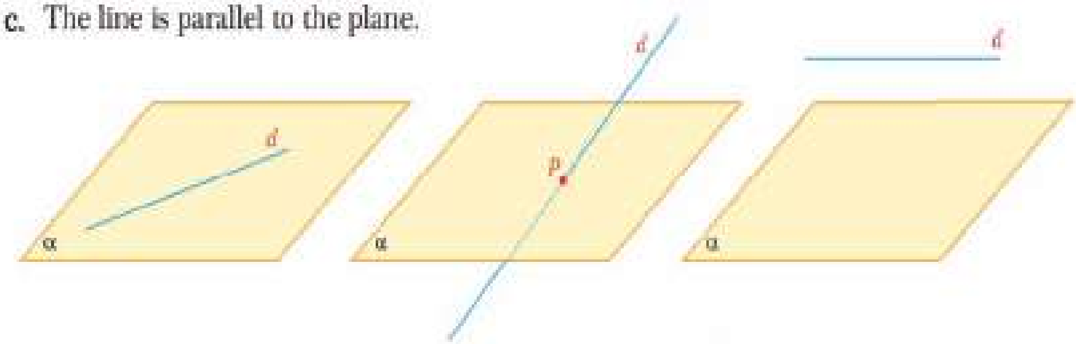
A line and a plane in space can have the following three positions relative to each other:

a. The line lies in the plane.

b. The line intersects the plane at a unique point.

c. The line is parallel to the plane.





Line d is lying in plane α

Line d intersects plane α at a unique point P

Line d is parallel to plane α

We can write the following theorems and conclusions concerning the position of a line relative to a plane:

Theorems

1. If a line d not in a plane is parallel to another line lying in the plane, line d will be parallel to the plane.
2. If a line is parallel to a plane, in this plane there are lines parallel to the given line.
3. Two lines parallel to the same line are parallel.
4. Two angles with corresponding parallel arms in the same direction are congruent.
5. If one of two parallel lines intersects a plane, the other line intersects it also.

Proofs 1. Let d be a line parallel to another line m lying in plane α as shown opposite.

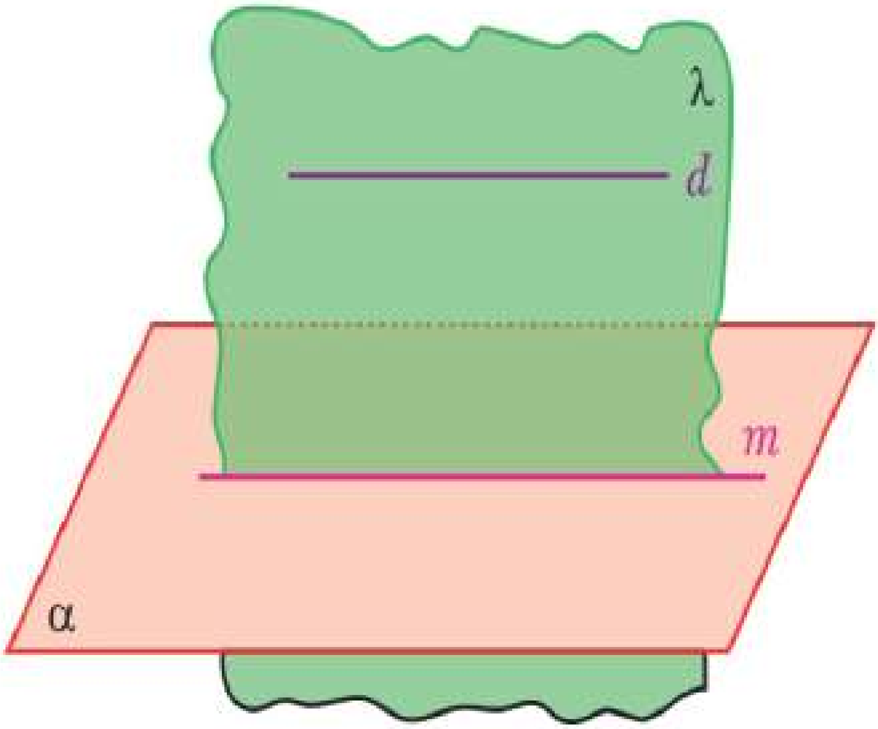
Since d and m are parallel lines they determine a plane λ by Rule 2 for the determination of a plane.

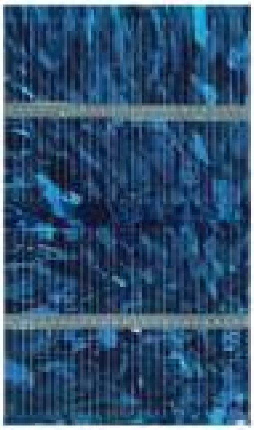
α and λ are intersecting planes along line m by Axiom 5.

If d and α intersect each other, their intersection point must be on m by Axiom 5.

So m and d intersect, i.e. they are not parallel. This is a contradiction.

In conclusion, d and α have no common point, i.e. they are parallel.





2. Look at the figure. Let d be a line parallel to a given plane α and let A be any point in plane β.

   Then d and A determine a plane β by Rule 3 for the determination of a plane.

   β and α have a common point, that is A. So they have a common line by Axiom 5.

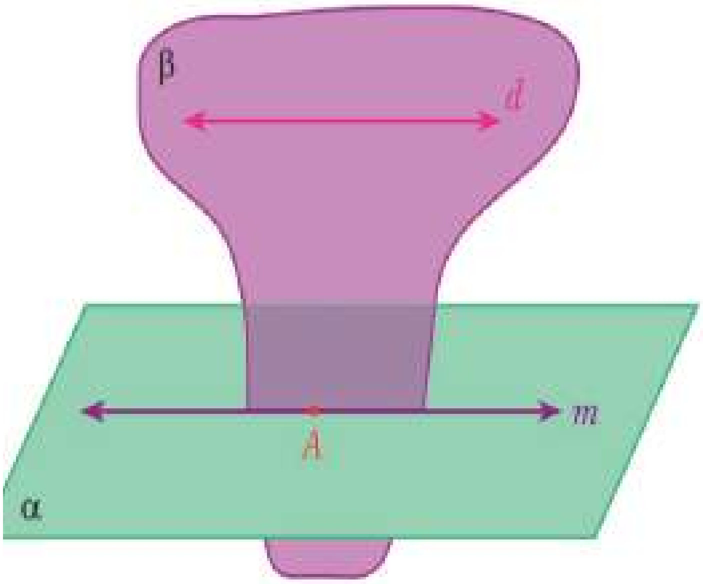
   Let us call this line m.

   Both d and m are in β.

   Since d has no common point with β, it cannot intersect m.

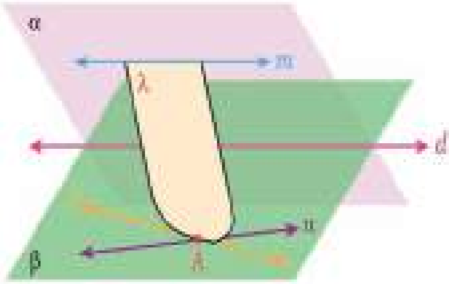
   So d and m are parallel lines by the rules of plane geometry.

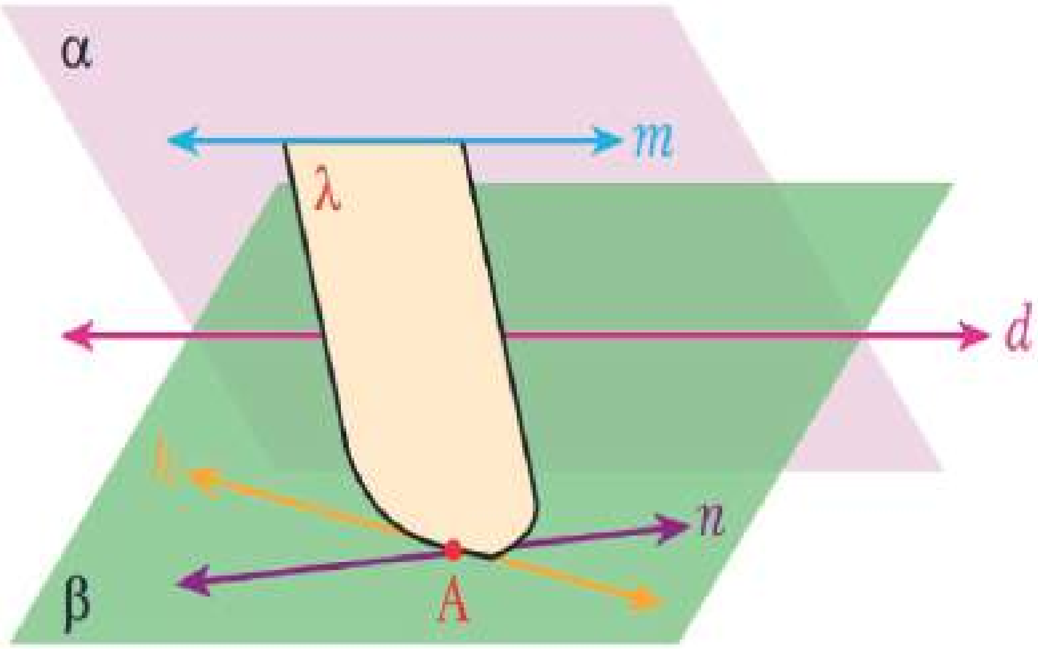
   We can find infinitely many parallel lines as there are infinitely many points A in the plane. We can conclude that if a line is parallel to a plane then in this plane there are infinitely many lines parallel to the given line, and these lines are parallel to each other.



3. Let m, n and d be three lines in space such that $m \parallel d$ and $n \parallel d$.

   Since m and d are parallel they determine a plane α, and since n and d are parallel they determine another plane β by Rule 2 for the determination of a plane.





Let A be a point on n.

Line m and point A determine a plane λ.

Since β and λ have a common point A, they have a common line k by Axiom 5.

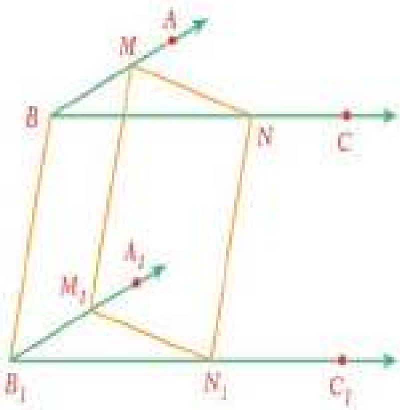
We know that $m // d$. So, m is parallel to β by the Theorem 1.

Then lines d and k are two lines in plane β parallel to line m. So k and d are parallel.

By the rules of plane geometry, we can draw only one line through point A which is parallel to d. So n and k must be coincident lines.

Therefore, m and n are parallel lines.

4. Let $\angle ABC$ and $\angle A_1B_1C_1$ be two angles with corresponding parallel arms in the same direction.



Let M and N be any two points on arms BA and BC respectively.

On B_1A_1 and B_1C_1 take two points M_1 and N_1 such that $M_1B_1 = MB$ and $N_1B_1 = NB$.

Since $BA // B_1A_1$, BMM_1B_1 is a parallelogram.

So $BB_1 // MM_1$ and $BB_1 = MM_1$. (1)

Similarly $BC // B_1C_1$ and BNN_1B_1 is a parallelogram.

So $BB_1 // NN_1$ and $BB_1 = NN_1$. (2)

From (1) and (2) we get $NN_1 // MM_1$ and $NN_1 = MM_1$.

So MNN_1M_1 is a parallelogram and $MN = M_1N_1$.

Then by S.S.S, ΔMBN and $\Delta M_1B_1N_1$ are congruent.

This means $\angle MBN \equiv \angle M_1B_1N_1$ and $\angle ABC \equiv \angle A_1B_1C_1$.

5. Let α be a plane and let d and m be two parallel lines such that d intersects α.

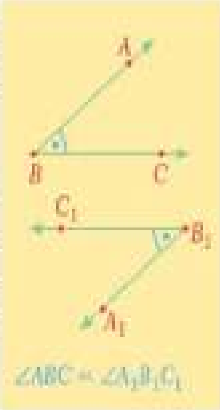
There are three possible cases: m lies in α or m is parallel to α or m intersects α.

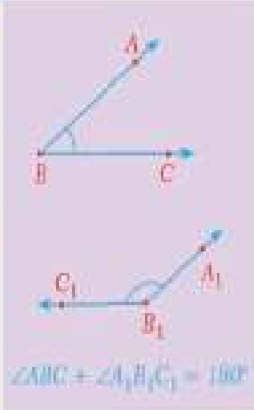
i. If m is in α then d will be parallel to a line in α. So d is parallel to α. This is a contradiction.

ii. If m is parallel to α then in α there will be a line (for example n) parallel to m. Since $d // m$ and $m // n$, we can conclude that $d // n$ by the Theorem 3. In this case again d will be parallel to α, which is a contradiction. Therefore m intersects α.

Corollary

- If one of two parallel lines is parallel to a plane then the other line is either in the plane or parallel to the plane.
- If the corresponding arms of two angles are parallel and in opposite directions, the angles are equal.
- If the corresponding arms of two angles are parallel and if one pair of corresponding arms is in the same direction while the other pair is in the opposite direction then the sum of the angles is 180°.





3. Relative Position of Two Planes

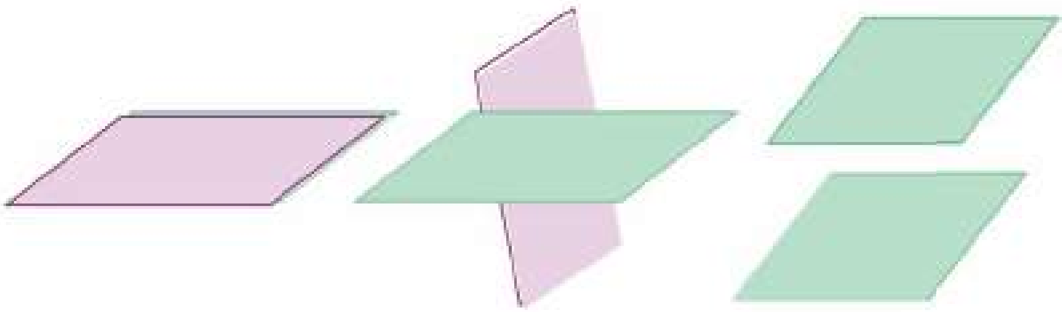
Rule

In space, there are three possible cases for the position of two planes relative to each other.

a. The planes can be coincident.

b. The planes can intersect.

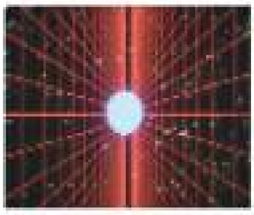
c. The planes can be parallel.



Theorems

1. If a plane passes through a line parallel to another plane and also intersects that plane then the line of intersection of the two planes is parallel to the given line.
2. If two intersecting lines in a plane are correspondingly parallel to two intersecting lines in another plane, the planes are parallel.
3. Through a point not in a plane we can draw one and only one plane parallel to the given plane.
4. If a line intersects one of two parallel planes, it intersects the other plane too.

Proofs

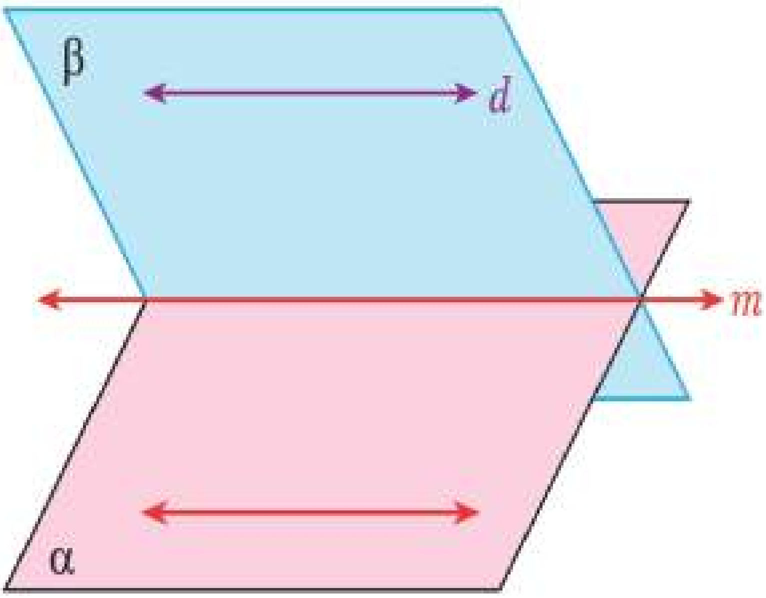


1. Let d be a line parallel to a plane α, and let β be a plane containing d and intersecting α along line m.

   Then d and m lie in β.

   Since m is in α and $d \parallel \alpha$, d and m cannot intersect each other. They are also not skew because they lie in the same plane.

   Therefore they are parallel.



2. Look at the figure. Let α and β be two planes.

Let m, n be two intersecting lines in α and let m_1, n_1 be two intersecting lines in β such that $m \parallel m_1$ and $n \parallel n_1$.

We need to prove that α and β are parallel, i.e. that they do not have any common point.

Let us assume that they have a common point and look for a contradiction.

If α and β have a common point then they will have a common line: their line of intersection. Let d be this line.

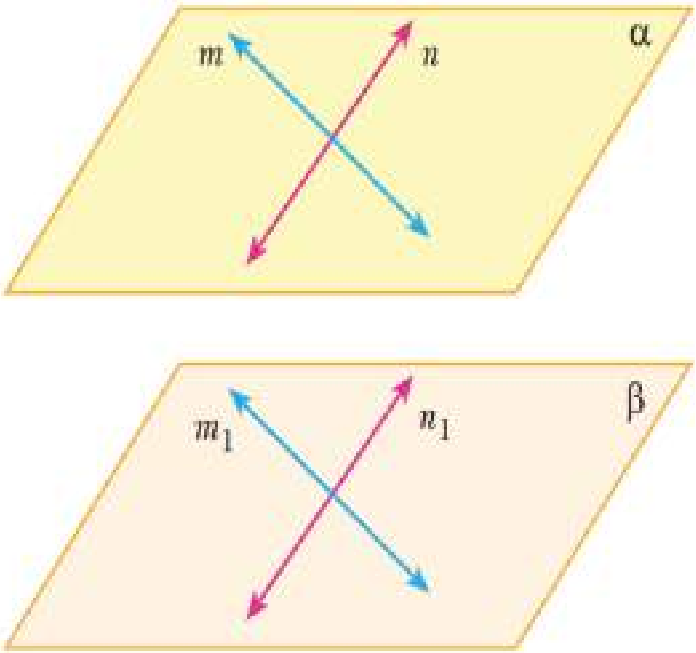
Since m and n are parallel to m_1 and n_1 respectively, both m and n are parallel to β.

$d \in \beta$, so neither m nor n can intersect line d.

As a result, since m, n and d are in the same plane, we must have $m \parallel d$ and $n \parallel d$.

But in this case, m and n must be coincident or parallel lines. This is a contradiction because it is given that they are intersecting lines.

This means that α and β do not have any common point. Therefore they are parallel planes.



3. Let α be a plane and let A be a point not in α.

We need to prove that

a. through A, we can draw a plane parallel to α, and

b. this plane is unique.

a. First we prove that the plane exists:

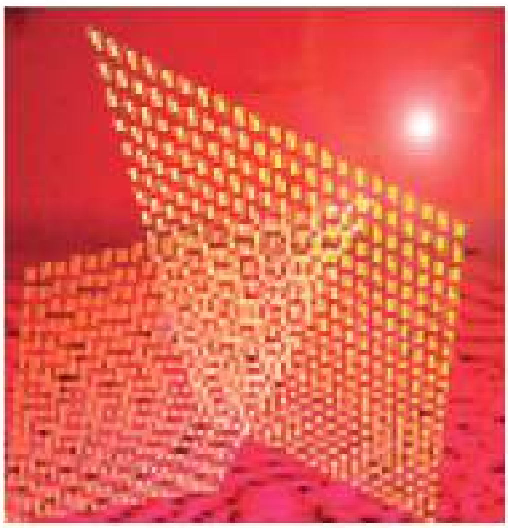
Let d and m be two intersecting lines in α.

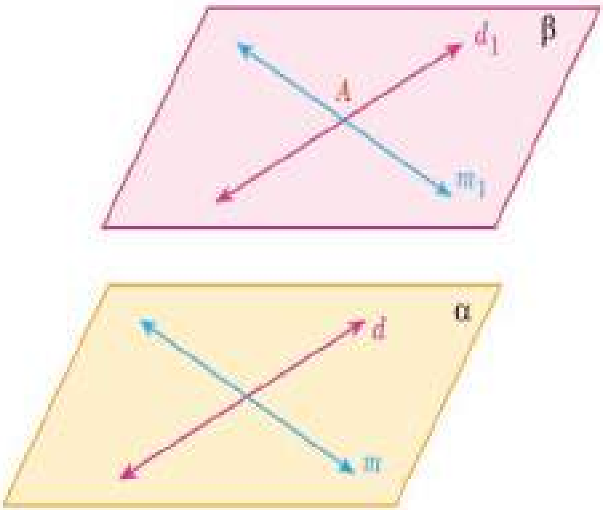
Through A we can draw a line parallel to d and another line parallel to m.

Let us name these lines d_1 and m_1.

Lines d_1 and m_1 are intersecting lines, so they determine a plane β.

By the previous theorem, α and β are parallel.





b. Now we prove that the plane is unique:

Assume that there is another plane β' containing A and parallel to α, and look for a contradiction.

β' cannot contain both d_1 and m_1, otherwise it would be coincident with plane β.

So at least one of d_1 and m_1 intersects β'. Let d_1 be this line.

Since d and d_1 are parallel, d also intersects plane β'.

This mean that β' cannot be parallel to α, which is a contradiction.

So plane β is unique.

4. Let α and β be two parallel planes and let d be a line intersecting α at a point A as shown in the figure.

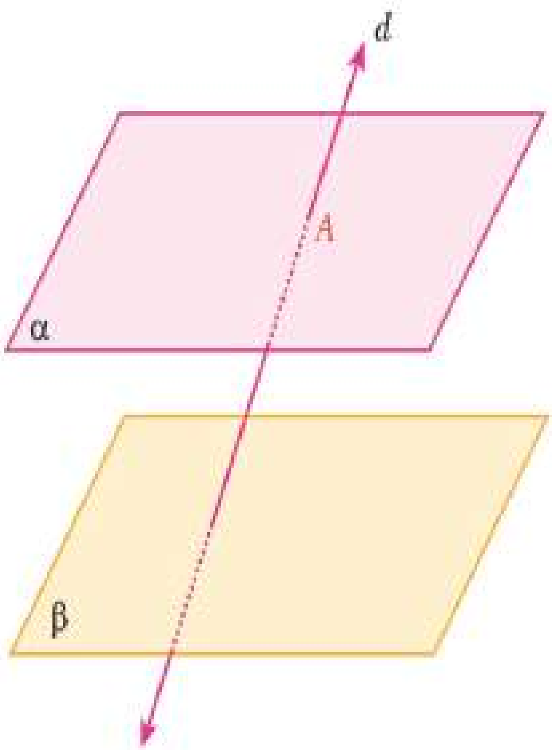
We need to prove that d intersects β.

Line d cannot lie in β because d intersects α and α // β.

Any line drawn through A and parallel to β must lie in α, so if d is parallel to β it lies in α.

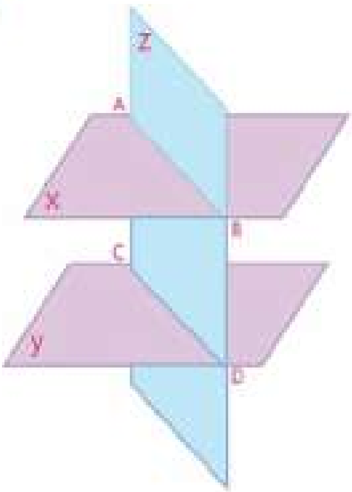
However, we know that d is not in α.

Hence there is only one possibility: line d is not parallel to β, i.e. it intersects β.



Theorems 1 If two plane are intersected by a third plane then the resulting intersection lines are parallel to each other

Proofs



$\left.\begin{array}{l}(X)\cap(Y)=\overleftrightarrow{AB}\\ (Y)\cap(Z)=\overleftrightarrow{CD}\end{array}\right\}$ *(given)*

$\left.\begin{array}{l}\overleftrightarrow{AB}\subset(X)\ ,\quad \overleftrightarrow{AB}\subset(Z)\\ \overleftrightarrow{CD}\subset(Y)\ ,\quad \overleftrightarrow{CD}\subset(Z)\end{array}\right\}$ *(intersection line of two planes contains all common points of planes)*

In (Z), if $\overleftrightarrow{AB}\nparallel\overleftrightarrow{CD}$ it intersects at a point E

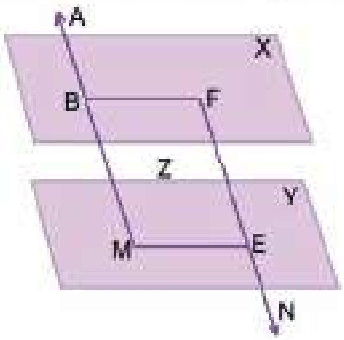
$\left.\begin{array}{l}E\in\overleftrightarrow{AB}\subset(X)\longrightarrow E\in(X)\\ E\in\overleftrightarrow{CD}\subset(Y)\longrightarrow E\in(Y)\end{array}\right\}$ *(intersection line of two planes contains all common points of planes)*

$E\in(X)\cap(Y)$ *they have common point E*

This is contrary to the given $(X)//(Y)$

Therefore $\overleftrightarrow{AB}$ does not intersect $\overleftrightarrow{CD}$ (Two lines are parallelif then either lie n a same plane or they are intersected)

Corollary of Theorems If a line intersects one of two parallel planes then it intersects the other plane too



Proofs

Let $E \in (Y)$ we draw $\overleftrightarrow{EN} // \overleftrightarrow{AB}$ *(One and only one straight line can be drawn parallel to a given straight line through a point outsite the given line)* (Parallel's Postulate)

Let (Z) contains $\overleftrightarrow{EF}$ and $\overleftrightarrow{AB}$ *(There is one and only one plane contains two parallel lines)*

$\overleftrightarrow{FB} // \overleftrightarrow{EM}$ (Theorem 1)

$\overleftrightarrow{AB}$ intersects (Y) at M

Theorems 2 If a plane contains one of two parallel lines then the plane is parallel to the other line

Proofs

If $\overleftrightarrow{AB}$ is not parallel to (X) so it intersects it at a given point E

$\overleftrightarrow{AB} // \overleftrightarrow{CD}$

(X) intersects $\overleftrightarrow{CD}$ *(A plane which intersects one of two parallel lines then it intersects the other line too)*

This is the contrary to the given$\left(\overleftrightarrow{CD} \subset (X)\right)$

(X) does not intersect $\overleftrightarrow{AB}$

$(X) // \overleftrightarrow{AB}$

Theorems 3 If two distinct lines are each parallel to a third line in a space then they are parallel to each other

Proofs

Let $A \in \overleftrightarrow{K}$

Let (X) contains $\overleftrightarrow{L}$ and the point A *(There is a unique plane contains a line and a point outside of the line)*

If $\overleftrightarrow{K} \not\subset (X)$, So $\overleftrightarrow{K}$ will intersect (X) at a point A

- (X) intersect $\overleftrightarrow{R}$, This is imposible *(A plane which intersects one of two parallel lines it intersects the other line too)*
- $\overleftrightarrow{K} \subset (X)$

  In (X) $\overleftrightarrow{K} \nparallel \overleftrightarrow{L}$

  $\Rightarrow \overleftrightarrow{K}$ intersect $\overleftrightarrow{L}$ at a point M

  So we have two straight lines drawn from the point M each parallel to $\overleftrightarrow{R}$

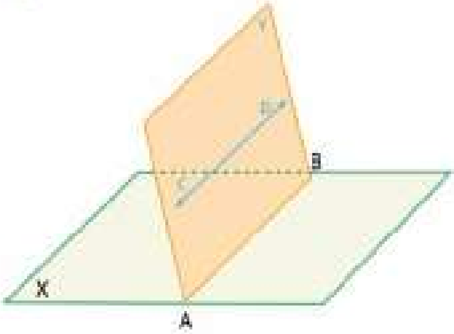
  This is contrary to given (Parallel's Postulate)

  $\overleftrightarrow{K}$ does not intersect $\overleftrightarrow{L}$

  $\overleftrightarrow{K} // \overleftrightarrow{L}$

Theorems 4

The intersection line of two planes is parallel to any line which is contained on one of the planes and parallel to the other



$\overleftrightarrow{AB}$, $\overleftrightarrow{CD} \subset (X)$
$\overleftrightarrow{CD} // (X)$ } Given

In (Y) , suppose $\overleftrightarrow{CD}$ intersects $\overleftrightarrow{AB}$

$\overleftrightarrow{CD}$ intersects (X) (The intersection line resulting from two intersected planes contains all their common points)

This is contrary to given $\overleftrightarrow{CD} // (X)$

$\overleftrightarrow{AB} // \overleftrightarrow{CD}$

Corollary of Theorems

If a plane is parallel to a plane and if a line is drawn from a point within the plane parallel to the given line then the line drawn lies in the plane

Proofs

If $\overleftrightarrow{CD} \not\subset (X)$

$\Rightarrow$ (X) intersects $\overleftrightarrow{CD}$ at C

$\overleftrightarrow{CD} // \overleftrightarrow{AB}$ (Given)

$\Rightarrow$ (X) intersects $\overleftrightarrow{AB}$ (A plane which intersects one of two parallel lines it intersects the other line to)

This is contrary to the given that

$\overleftrightarrow{AB} // (X)$

$\Rightarrow \overleftrightarrow{CD}$ does not intersect (X)

Example 1 If two intersected planes each contains one of two parallel lines then the intersection line of the planes is parallel to each of the parallel lines

$\left.\begin{array}{l}\overleftrightarrow{AB} // \overleftrightarrow{CD} \\ \overleftrightarrow{CD} \subset (Y)\end{array}\right\}$ Given

$\overleftrightarrow{AB} // (Y)$ (Theorem 2)

$\overleftrightarrow{AB} \subset (X)$ (Given)

$\overleftrightarrow{AB} // \overleftrightarrow{EF}$ (Theorem 4)

$\Rightarrow \overleftrightarrow{CD} // \overleftrightarrow{EF}$ (Theorem 3)

Perpendicularity of Lines and Planes

1. A line is perpendicular to two intersecting lines at their intersected point if it is perpendicular to their plane
2. A line is perpendicular to a plane if it is perpendicular to all lines in the plane through its trace
3. There is one and only one straight line can be drawn perpendicular to a given plane from a given point

   Let C be a point, so either $C \notin (x)$ or $C \in (X)$

   There is a unique line $\overleftrightarrow{L}$ Passes through the point C such that $\overleftrightarrow{L} \perp (x)$
4. The line $\overleftrightarrow{AB}$ is inclined (oblique) to the plane if it intersect the plane and it is not perpendicular to it.

   If $\overleftrightarrow{AB} \cap (X) = \{C\}$ and $\overleftrightarrow{AB}$ is not perpendicular to (X) then $\overleftrightarrow{AB}$ is inclined to (X)

   Note: $\overleftrightarrow{AB}$ is not perpendicular to (X) if $\overleftrightarrow{AB}$ is inclined or parallel to (X)
5. The length of a line segment drawn perpendicular to a plane from a given point to the trace is called the distance of the point to the plane

   AB is the distance of the point A to (X)
6. The Length of the line segment bounded by two parallel planes and drawn perpendicular to the planes is called the distance between two planes

   If $(X) // (Y), \overleftrightarrow{AB} \perp (X), \overleftrightarrow{AB} \perp (X)$ Then AB represents the distance between (X) , (Y)
7. If a line is perpendicular to one of two parallel planes it is perpendicular to the other plane too.
8. Two planes are parallel if they are each perpendicular to a straight line

Theorems 5 If a plane is perpendicular to one of two parallel lines then it is perpendicular to the other line too.

Proofs

$\overleftrightarrow{CD} \cap (X) = \{D\}$ (Theorem 1)

In (X) we draw $\overleftrightarrow{BE}, \overleftrightarrow{BF}$

And $\left.\begin{matrix} \overleftrightarrow{DG} // \overleftrightarrow{BE} \\ \overleftrightarrow{DH} // \overleftrightarrow{BF} \end{matrix}\right\}$ (Parallel's Postulate)

$\left.\begin{matrix} m\sphericalangle ABE = m\sphericalangle CDG \\ m\sphericalangle ABF = m\sphericalangle CDH \end{matrix}\right\}$

(If each two sides of an angle is parallel to two sides of another angle then their measurements of the angles are equal and their planes are parallel)

$\overleftrightarrow{AB} \perp (X)$ (given)

$\overleftrightarrow{AB} \perp \overleftrightarrow{BE}, \overleftrightarrow{BF}$

$m\sphericalangle ABE = m\sphericalangle CDG = 90$

$m\sphericalangle ABF = m\sphericalangle CDH = 90$

$\overleftrightarrow{CD} \perp (X)$

Corollary of Theorems If two lines are perpendicular to a plane then the lines are parallel.

Proofs

If $\overleftrightarrow{AB}$ is not parallel to $\overleftrightarrow{CD}$ then

From $D \in (X)$ we draw $\overleftrightarrow{DE} // \overleftrightarrow{AB}$ (Parallel's Postulate)

$\overleftrightarrow{AB} \perp (X)$ (given)

$\overleftrightarrow{DE} \perp (X)$ (theorem 5)

$\overleftrightarrow{CD} \perp (X)$ (given)

There are two lines both drawn from $\overleftrightarrow{D}$ perpendicular to (X) it is ipmossible, because there is one one and only one line can be drawn perpendicular from a point to a given plane

$\overleftrightarrow{DE} = \overleftrightarrow{DC}$

$\overleftrightarrow{AB} // \overleftrightarrow{CD}$

Theorems 5 (Theorem of the line perpendicular)

If two lines are drawn from a point in a plane one of them perpendicular to the plane and the other perpendicular to a given line within the plane, then the join line between any point of perpendicular line to a plane and the point of intersected lines is perpendicular to the given line in the plane.

Proofs

From point B we draw $\overleftrightarrow{BF} // \overleftrightarrow{CD}$ (Parallel's Postulate)

$\overleftrightarrow{CD} \subset (X)$ (given)

$\Rightarrow \overleftrightarrow{BF} \subset (X)$ (if two lines are parallel then the plane which contains one of the two lines and a point of the other line contains the two lines)

$\overleftrightarrow{BE} \perp \overleftrightarrow{CD}$ (given)

$\overleftrightarrow{BF} \perp \overleftrightarrow{BE}$ (in a plane, a line which is perpendicular to one of two parallel lines is perpendicular to the other line too)

$\overleftrightarrow{AB} \perp (X)$ (given)

$\overleftrightarrow{NB} \perp \overleftrightarrow{BF}$

$\overleftrightarrow{BF} \perp (NBE)$

$\overleftrightarrow{CD} \perp (NBE)$ (Theorem 5)

$\overleftrightarrow{EN} \perp \overleftrightarrow{CD}$

The same way we can prove that any line joining a point of $\overleftrightarrow{AB}$ and the point E, is perpendicular to $\overleftrightarrow{CD}$

Corollary of Theorems

If two lines are drawn from a point outside of al plane, one fo them perpendicular to the plane and the other line perpendicular to a given line in the plane, then the join line between the traces of two perpendicular lines is perpendicular to the given line in the plane

Proofs

If $\overleftrightarrow{BE}$ is not perpendicular to $\overleftrightarrow{CD}$ so from the point B we draw $\overleftrightarrow{NB} \perp \overleftrightarrow{CD}$ *(There is one and only one perpendicular line can be drawn to a given line from outside point)*

$\overleftrightarrow{AB} \perp (X)$ (given)

$\overleftrightarrow{AN} \perp \overleftrightarrow{CD}$ (Theorem 6)

$\overleftrightarrow{AE} \perp \overleftrightarrow{CD}$ (given)

$\overleftrightarrow{AN} \equiv \overleftrightarrow{AE}$ *(There is one and only one perpendicular line can be drawn to a given line from outside point)*

$N \equiv E$

$\overleftrightarrow{BE} \equiv \overleftrightarrow{BN}$

$\overleftrightarrow{BE} \equiv \overleftrightarrow{CD}$

SOLVED PROBLEMS

1) BCD right triangle at B, A is a point outside of triangle plane such that AC=CD , AB=BD. Prove that $\overleftrightarrow{BC}$ is perpendicular to a triangle ABD plane

Proofs

In triangles ABC,BCD

$\overleftrightarrow{AB} = \overleftrightarrow{BD}$ (given)

$\overleftrightarrow{AC} = \overleftrightarrow{CD}$ (given)

$\overleftrightarrow{BC}$ is common

$\triangle ABC \equiv \triangle BCD$

$m\sphericalangle CBD \equiv m\sphericalangle ABC = 90^\circ$

$\overleftrightarrow{BC} \perp \overleftrightarrow{BD}$ $(m\sphericalangle BCD = 90^\circ)$ (given)

$\overleftrightarrow{BC} \perp \overleftrightarrow{AB}$ $(m\measuredangle ABC = 90^{\circ})$ (proved)

$\overleftrightarrow{BC} \perp (ABD)$

1) $\overleftrightarrow{AB}$ is a diameter of a circle whose center is O, c is a point on the circle, $\overleftrightarrow{CD}$ is perpendicular to circle's plane. Prove that $\overleftrightarrow{AC} \perp (BCD)$

Proofs

$\overleftrightarrow{AB}$ is a diameter of the circle (Given)

$m\measuredangle ACB = 90^{\circ}$ *(A circumference angle drawn in a half circle is right right angle)*

$\overleftrightarrow{AC} \perp \overleftrightarrow{BC}$

$\overleftrightarrow{CD} \perp (ABC)$ (Given)

$\overleftrightarrow{AC} \perp \overleftrightarrow{CD}$

$\overleftrightarrow{AC} \perp (BCD)$

2) ABC is right angle triangle at B, $\overline{AE} \perp (ABC)$, D is midpoint of $\overline{CE}$, N is midpoint of $\overline{AB}$. Prove that $\overline{AB} \perp \overline{ND}$

Proofs

Let M be a midpoint of $\overline{AC}$

D is midpoint of (given)

N is midpoint of (given)

$\overline{MD} // \overline{AE}$ (The segment line joining between two midpoints of two sides of a triangle is parallel to the third side)

$\overline{MN} // \overline{BC}$ (The same reason)

$\overline{AE} // (BCD)$ (given)

$\overline{MD} // (BCD)$ (Theorem 5)

$\measuredangle B$ is right angle (given)

$\Rightarrow \overline{AB} \perp \overline{BC}$

$\overline{MN} \perp \overline{AB}$ (in a plane if a line is perpendicular to one of two parallel lines, it is parallel to the other too)

$M \in (ABC)$

$\Rightarrow \overline{MD} \perp (ABC)$, $\overline{MN} \perp \overline{AB}$, $\overline{AB} \subset (ABC)$

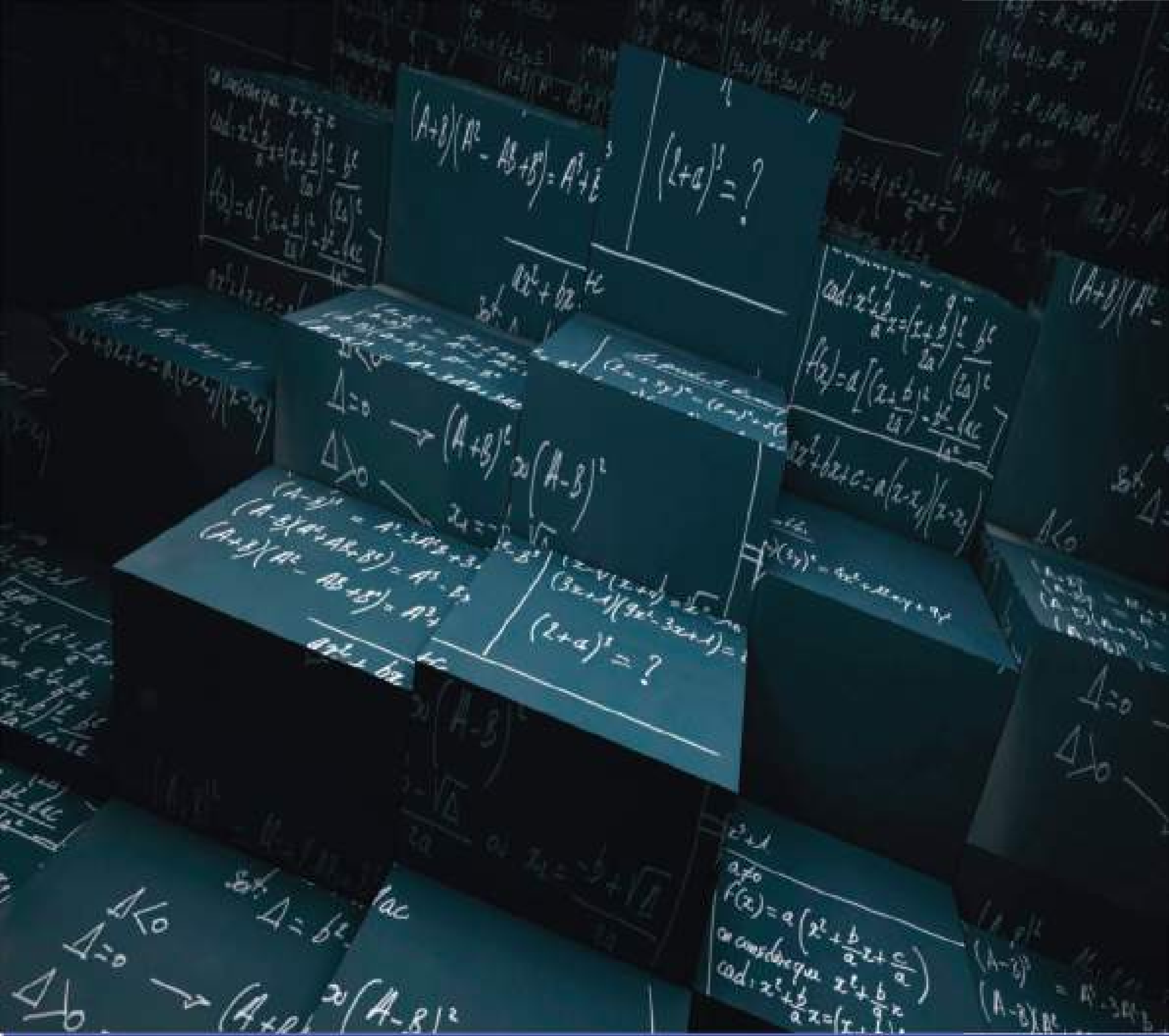
$\Rightarrow \overline{AB} \perp \overline{ND}$ (Theorem of perpendicular)

EXERCISES

1) ABC is right angle triangle at B, AB=4cm, BC=3cm, $\overline{CD} \perp (ABC)$ such that CD=12cm, Find the length of $\overline{AD}$

2) Prove that if two lines are each perpendicular to two intersected planes then the lines are not parallel.

3) In ΔABC, $m\measuredangle A = 30^{\circ}$, $\overline{BD} \perp (ABC)$, $BD = 5cm$, $AB = 10cm$, if $\overline{BH} \perp \overline{AC}$ then find $m\measuredangle BHD$



Chapter 8

PERMUTATION
COMBINATION
PROBABILITY

PERMUTATIONS

We can define a **permutation** as an ordered arrangement of some or all of the elements in a given set. The way a set of books is arranged on a shelf, the seating positions of a group of people at a table or the way the players in a football team line up for a team photo are some examples of permutations since in each case, the order of the elements is important.

A. FACTORIAL NOTATION

When we are solving permutation problems, we often need to express the product of all consecutive counting numbers from 1 to a number n. **Factorial notation** provides an easy way to denote this product.

Definition

factorial

For any counting number n, the product of all positive integers less than or equal to n is called **n factorial** and denoted by $n!$:

$$n! = n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1.$$

For example, $3! = 3 \cdot 2 \cdot 1 = 6$ and $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.

As a special case we accept that $0! = 1$.

EXAMPLE 1 Evaluate the expressions.

a. $7!$ b. $6! - 3!$ c. $(6-3)!$ d. $4! + 2!$

e. $(4+2)!$ f. $\frac{8!}{4!}$ g. $\left(\frac{8}{4}\right)!$

Solution

a. $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$

b. $6! - 3! = (6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) - (3 \cdot 2 \cdot 1) = 720 - 6 = 714$

c. $(6-3)! = (3)! = 3! = 3 \cdot 2 \cdot 1 = 6$

d. $4! + 2! = (4 \cdot 3 \cdot 2 \cdot 1) + (2 \cdot 1) = 24 + 2 = 26$

e. $(4+2)! = (6)! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$

f. $\frac{8!}{4!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 8 \cdot 7 \cdot 6 \cdot 5 = 1680$

g. $\left(\frac{8}{4}\right)! = (2)! = 2 \cdot 1 = 2$

Remark

For all positive integers, $n! = n(n-1)!$

For example, $7! = 7 \cdot \underbrace{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}_{6!} = 7 \cdot 6!$.

As a result of this property, we can write
$n! = n(n-1)! = n(n-1)(n-2)! = n(n-1)(n-1)(n-2)(n-3)!$, etc.

EXAMPLE 2

Evaluate the expressions.

a. $\dfrac{9!}{8!}$ b. $\dfrac{13!}{11!}$ c. $100! - 99!$ d. $\dfrac{10!+8!}{9!-7!}$

Solution

a. $\dfrac{9!}{8!} = \dfrac{9 \cdot 8!}{8!} = 9!$

b. $\dfrac{13!}{11!} = \dfrac{13 \cdot 12 \cdot 11!}{11!} = 156$

c. $100! - 99! = (100 \cdot 99!) - 99! = 99!(100 - 1) = 99 \cdot 99!$

d. $\dfrac{10!+8!}{9!-7!} = \dfrac{(10 \cdot 9 \cdot 8 \cdot 7!)+(8 \cdot 7!)}{(9 \cdot 8 \cdot 7!)-7!} = \dfrac{7!(10 \cdot 9 \cdot 8+8)}{7!(9 \cdot 8-1)} = \dfrac{728}{71}$

EXAMPLE 3

Simplify the expressions.

a. $\dfrac{n!}{(n-1)!}$ b. $\dfrac{(n+3)!}{(n-1)!} \cdot \dfrac{(n-2)!}{(n+2)!}$ c. $\dfrac{(n+1)!}{n \cdot (n-2)!}$

Solution

a. $\dfrac{n!}{(n-1)!} = \dfrac{n(n-1)!}{(n-1)!} = n$

b. $\dfrac{(n+3)!}{(n-1)!} \cdot \dfrac{(n-2)!}{(n+2)!} = \dfrac{(n+3) \cdot (n+2)!}{(n-1) \cdot (n-2)!} \cdot \dfrac{(n-2)!}{(n+2)!} = \dfrac{(n+3)}{(n-1)}$

c. $\dfrac{(n+1)!}{n \cdot (n-2)!} = \dfrac{(n+1) \cdot (n-1) \cdot n \cdot (n-2)!}{n \cdot (n-2)!} = (n+1) \cdot (n-1) = n^2 - 1$

EXAMPLE 4 Solve $\frac{(n+2)!}{(n^3-n)\cdot(n-3)!}=6n-12.$

Solution

$$\frac{(n+2)!}{(n^3-n)(n-3)!}=\frac{(n+2)\cdot(n+1)\cdot n\cdot(n-1)\cdot(n-2)\cdot(n-3)!}{n(n^2-1)(n-3)!}$$

$$=\frac{(n+2)\cdot(n-2)\cdot \cancel{n}\cdot\cancel{(n+1)}\cdot\cancel{(n-1)}\cdot\cancel{(n-3)!}}{\cancel{n}\cdot\cancel{(n+1)}\,\cancel{(n-1)}\,\cancel{(n-3)!}}$$

$$=(n+2)\cdot(n-2)=n^2-4.$$

So $n^2-4=6n-12$, which gives $n^2-6n+8=0$.

This equation has two roots: $n_1=2$ and $n_2=4$.

Since the first root makes $(n-3)!$ invalid, $n=4$.

Remember!
In the factorial expression $n!$, n must be a counting number.

Check Yourself

1. Evaluate the expressions.

a. $\frac{13!}{10!3!}$ b. $\frac{15!-14!}{15!+14!}$ c. $\frac{(n+2)!(n-1)!}{(n+1)!n!}$

2. Solve for n.

a. $\frac{(n+1)!}{(n-1)!}\cdot\frac{(n-2)!}{n!}=\frac{7}{5}$ b. $n(n^2-1)(n-2)!=720$

Answers

1. a. 286 b. $\frac{7}{8}$ c. $\frac{n+2}{n}$ 2. a. 6 b. 5

B. PERMUTATIONS OF *r* ELEMENTS SELECTED FROM *n* ELEMENTS

Many permutation problems ask us to consider arrangements of r things chosen from n things ($0 \le r \le n$), i.e. permutations of r elements chosen from a set of n elements.

$$P(n,n) = \frac{n!}{(n-n)!}$$

EXAMPLE 5 How many different two-letter combinations can we form from the letters of the word KANO if a letter cannot be used more than once?

Solution The order of the letters is important and a letter cannot be used more than once. By the multiplication principle, the number of combinations is: $\underline{4} \cdot \underline{3} = 12$. These combinations are

KA	AK	NK	OK
KN	AN	NA	OA
KO	AO	NO	ON

.

In this section we will use a new formula to solve problems of this type.

Definition permutation of r elements selected from n elements

The number of permutations of r elements selected from a set of n elements is

Some books use ${}_nP_r$ or P_n^r to mean $P(n, r)$.

If we apply this formula to Example 39, we can write the answer as

Note

Any question which can be solved using this permutation formula can also be solved using the multiplication principle.

EXAMPLE 6 Calculate $P(5, 3) \cdot P(7, 2)$.

Solution

$$P(5,\ 3) \cdot P(7,\ 2) = \frac{5!}{(5-3)!} \cdot \frac{7!}{(7-2)!} = \frac{\cancel{5!}}{2!} \cdot \frac{7!}{\cancel{5!}} = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2520$$

EXAMPLE 7 Evaluate the expressions.

a. $P(7, 3)$ b. $P(n, n)$ c. $P(n, 0)$

Solution

a. $P(7,3) = \dfrac{7!}{(7-3)!} = \dfrac{7!}{4!} = \dfrac{7\cdot 6\cdot 5\cdot 4!}{4!} = 210$

b. $P(n,n) = \dfrac{n!}{(n-n)!} = \dfrac{n!}{0!} = \dfrac{n!}{1} = n!$

c. $P(n,0) = \dfrac{n!}{(n-0)!} = \dfrac{n!}{n!} = \dfrac{n!}{n!} = 1$

EXAMPLE 8 $P(n, 3)\cdot 5 = P(n, 4)$ is given. Find n.

Solution $P(n, 3)\cdot 5 = P(n, 4)$

$$\frac{n!}{(n-3)!}\cdot 5 = \frac{n!}{(n-4)!}$$

$$\frac{5}{(n-3)\cdot (n-4)!} = \frac{1}{(n-4)!}$$

$$5 = n-3$$

$$n = 8$$

EXAMPLE 9 How many three-digit numbers can be formed from the digits in the number 13567 if a digit cannot be used more than once?

Solution We are choosing three digits from five digits. So there are

$P(5,3) = \dfrac{5!}{(5-3)!} = \dfrac{5\cdot 4\cdot 3\cdot 2!}{2!} = 60$ different three-digit numbers.

Notice that we could have solved the same question using the multiplication principle: $5\cdot 4\cdot 3 = 60$.

EXAMPLE 10 Three raffle tickets will be selected in order from a box containing 30 tickets. The person holding the first ticket will win a car, the person with the second ticket will win a computer, and the person with the third ticket will win a CD player. In how many different ways can these prizes be awarded?

Solution Since the question is about an ordered arrangement of three tickets selected from thirty tickets, we can use the formula:

$P(30,3) = \dfrac{30!}{(30-3)!} = \dfrac{30!}{27!} = \dfrac{30\cdot 29\cdot 28\cdot 27!}{27!} = 24360.$

Remark

The number of permutations of r elements selected from n elements is the product of r successive numbers less than or equal to n:

$$P(n, r) = \underbrace{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1)}_{r \text{ factors}}.$$

For example,

$P(5, 4) = \underbrace{5 \cdot 4 \cdot 3 \cdot 2}_{4 \text{ factors}} = 120$, $P(10, 3) = \underbrace{10 \cdot 9 \cdot 8}_{3 \text{ factors}} = 720$ and $P(20, 1) = \underbrace{20}_{1 \text{ factor}} = 20$.

EXAMPLE 11 A fighter plane has seats for a pilot and a copilot. In how many different ways can these be selected from a squadron of 18 soldiers?

Solution $P(18, 2) = \underbrace{18 \cdot 17}_{2 \text{ factors}} = 306$

EXAMPLE 12 How many different combinations of at least 3 letters can be formed from the letters in the word MATHS if no letter can be used more than once?

Solution 'At least 3 letters' means the combination can have 3 letters, 4 letters or 5 letters. So we need to consider three mutually exclusive cases: combinations of 3 letters, 4 letters and 5 letters.

Mutually exclusive cases are cases which cannot happen at the same time.

Then we add the number of permutations in each case:

$$\underbrace{P(5,3)}_{3\text{-letter words}} + \underbrace{P(5,4)}_{4\text{-letter words}} + \underbrace{P(5,5)}_{5\text{-letter words}} = (5 \cdot 4 \cdot 3) + (5 \cdot 4 \cdot 3 \cdot 2) + (5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) = 60 + 120 + 120 = 300.$$

So there are 300 possible combinations.

EXAMPLE 13 Kemal's bookcase has three shelves. Kemal has 5 different math books, 6 different biology books and 7 different physics books. He wants to arrange 3 math books, 4 biology books and 5 physics books on the shelves so that each shelf is for one subject only. In how many different ways can Kemal arrange his books?

Solution There are $P(5, 3)$ possible ways to arrange the math books. There are also $P(6, 4)$ and $P(7, 5)$ different possible ways to order the biology and physics books respectively.

However, Kemal can choose the shelves for the subjects in 3! ways. As a result there are

$$\underbrace{P(5,3)}_{\text{math}} \cdot \underbrace{P(6,4)}_{\text{biology}} \cdot \underbrace{P(7,5)}_{\text{physics}} \cdot \underbrace{3!}_{\text{shelves}} = 60 \cdot 360 \cdot 840 \cdot 6$$

= 108 864 000 ways for Kemal to arrange his books.

EXAMPLE 14 A three-digit number is formed by choosing elements from the set $\{1, 3, 4, 5, 7, 8, 9\}$ without repetition.

a. How many numbers do not contain the digit 5?

b. How many numbers contain the digit 5?

c. How many numbers contain 1 or 7 or both 1 and 7?

Solution a. The problem is the same as finding the number of three-digit permutations of the set $\{1, 3, 4, 7, 8, 9\}$ (5 excluded): $P(6, 3) = 6 \cdot 5 \cdot 4 = 120$.

b. The total number of three-digit permutations of the set $\{1, 3, 4, 5, 7, 8, 9\}$ is $P(7, 3) = 7 \cdot 6 \cdot 5 = 210$. From part a, 120 of these permutations do not contain the digit 5. So there are $210 - 120 = 90$ three-digit numbers which contain the digit 5.

c. We begin by calculating the number of three-digit permutations in which 1 and 7 are not used: $P(5, 3) = 5 \cdot 4 \cdot 3 = 60$ permutations. So there are $210 - 60 = 150$ three-digit numbers which contain 1 or 7 or both 1 and 7.

Check Yourself

1. There are 7 different pieces of fruit on a tray. We will choose 3 of them and arrange them in a row on a plate. How many different arrangements are possible?
2. The students in a class are photographed in pairs such that each student is photographed with every other student. If there are 90 photos, how many students are there in the class?
3. A machine generates all the possible two-letter combinations of the letters *ABCDE*, without using a letter twice. What percentage of the combinations do not contain a consonant?
4. How many of the four-digit numbers formed from the digits of the number 12345 without repetition do not begin with the digit 2?
5. A group *A* contains 6 students and a group *B* contains 8 students. In a class photo, two students who are to sit in the front will be from *A* and three students who are to stand at the back will be from *B*. How many arrangements are possible?

Answers

1. $P(7, 3) = 210$ 2. 10 3. 10% 4. $P(5, 4) - P(4, 3) = 96$ 5. $P(6, 2) \cdot P(8, 3) = 10080$

EXERCISES

A. Factorial Notation

1. Write $24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19$ using factorial notation.

2. Evaluate $\dfrac{5!+7!}{5!+6!}$.

3. Evaluate $\dfrac{10!-2\cdot 8!}{(5!+2\cdot 3!)\cdot 7!}$.

4. Simplify the expressions.

   a. $\dfrac{n!}{(n-2)!}$ b. $\dfrac{(n+2)!+(n+1)!}{2\cdot n!+(n+1)!}$

   c. $\dfrac{(2n+2)!}{(2n-1)!}\cdot\dfrac{(n-1)!}{(n+2)!}$

5. Solve the equations.

   a. $\dfrac{(n+1)!}{n!}=17$ b. $\dfrac{(x-2)!}{(x-4)!}=18-x$

   c. $\dfrac{(n+1)!}{(n-1)!}=56$

6. Simplify $(1\cdot 1!)+(2\cdot 2!)+(3\cdot 3!)+\ldots+(n\cdot n!)$.

B. Permutations of *r* Elements Selected from *n* Elements

7. Evaluate the expressions.

   a. $P(11, 2)$ b. $P(8, 3)\cdot P(5, 4)$

   c. $\dfrac{P(n, 4)}{P(n, 3)}$ d. $\dfrac{P(4, 3)+P(8, 3)}{P(6, 3)}$

   e. $\dfrac{P(5, 5)}{P(7, 7)}$

8. How many different five-digit numbers can be formed by using the digits in the number 75491 once?

9. In how many ways can a group of 7 students be seated in a row of 7 chairs if a particular student insists on being in the first chair?

10. In how many different ways can we name a regular pentagon using letters *P*, *Q*, *R*, *S*, *T*?

11. Seven people will be in a group photograph. In how many different ways can the photograph be set up if 3 people must be in front and 4 must be at the back?

12. A group photograph will be taken of 5 boys and 5 girls. Five people must be in the front and 5 people must be at the back. If the girls must sit together, in how many ways can the photograph be taken?

COMBINATIONS

When the order of the elements chosen from a set is important, we use permutation. However, order is not always important when we are choosing elements. For example, we may want to choose a certain number of people from a group to form a committee. The order of the chosen members is not important since the result is a group of people, not an ordered set. An unordered selection of elements like this is called a **combination**.

An r-element subset is a subset with r elements.

When we talk about a combination of n objects taken r at a time, we mean the r-element subsets of a set with n elements. We write total the number of such combinations as

$$C(n, r) \text{ or } \binom{n}{r} \quad (n, r \in \mathbb{Z} \text{ and } 0 \leq r \leq n).$$

For example, if we are asked to choose two digits from the set $\{2, 3, 5\}$, we might choose $\{3, 5\}$ or $\{5, 3\}$. These are the same combination. This is very different to the problem of forming a two-digit number using the digits 3 and 5 because 35 and 53 are two different outcomes.

A. COMBINATIONS OF *r* ELEMENTS SELECTED FROM *n* ELEMENTS

Consider the set $K = \{1, 2, 3\}$. Let us compare the two-element combinations with the two-element permutations of the set K in a table:

$K = \{1, 2, 3\}$		
Combinations with 2 elements	Permutations with 2 elements	
$\{1, 2\}$	12	21
$\{2, 3\}$	23	32
$\{1, 3\}$	13	31

We can see that the number of permutations with two elements is twice the number of the combinations with two elements: $2 \cdot C(3, 2) = P(3, 2)$.

If we now consider the three-element combinations and permutations of the set $A = \{a, b, c, d\}$, we get the following table:

$A = \{a, b, c, d\}$						
Combinations with 3 elements	Permutations with 3 elements					
$\{a, b, c\}$	abc	acb	bac	bca	cab	cba
$\{a, b, d\}$	abd	adb	bad	bda	dab	dba
$\{a, c, d\}$	acd	adc	cad	cda	dac	dca
$\{b, c, d\}$	bcd	bdc	cbd	cdb	dbc	dcb

There are four combinations and 24 permutations. We can see that the number of permutations with three elements is 3! times the number of combinations with three elements: $3! \cdot C(4, 3) = P(4, 3)$.

If we repeated this exercise for two-element permutations and combinations we would find $2! \cdot C(4, 2) = P(4, 2)$.

We can generalize this pattern as

$\underbrace{C(n, r)}_{\substack{\text{ways of choosing} \\ \text{a group} \\ \text{with } r \text{ elements}}} \cdot \underbrace{r!}_{\substack{\text{ways of arranging} \\ \text{arranging those} \\ \text{elements}}} = P(n, r)$, which gives us the formula

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!/(n-r)!}{r!} = \frac{n!}{(n-r)! \cdot r!}.$$

Definition

combination

Let n and r be non-negative integers such that $0 \le r \le n$.

A subset of r elements chosen from a set of n elements is called an **r-element combination** of that set.

The number of r-element combinations of a set of n elements is

$$C(n, r) = \frac{n!}{r! \cdot (n-r)!} \quad (n, r \in \mathbb{Z} \text{ and } 0 \le r \le n).$$

$C(n, r)$ is sometimes written as C^n_r, $C\binom{n}{r}$, ${}_nC_r$ or ${}^nC_r$.
${}_nC_r$ is sometimes read as 'n, choose r'.

EXAMPLE 15 Calculate $C(8, 3)$.

Solution By the formula, $C(8,3) = \frac{8!}{3! \cdot (8-3)!} = \frac{8!}{3! \cdot 5!} = \frac{8 \cdot 7 \cdot 6 \cdot 5!}{3! \cdot 5!} = 56.$

EXAMPLE 16 Evaluate $C(12, 5) \cdot C(7, 2)$.

Solution

$$C(12, 5) \cdot C(7, 2) = \frac{12!}{5! \cdot (12-5)!} \cdot \frac{7!}{2!(7-2)!} = \frac{12!}{5! \cdot 7!} \cdot \frac{7!}{2! \cdot 5!} = \frac{12!}{5! \cdot 2! \cdot 5!}$$

$$= \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5!}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 2 \cdot 5!} = 11 \cdot 2 \cdot 9 \cdot 2 \cdot 7 \cdot 6 = 16632$$

EXAMPLE 17 Find the number of groups of 3 students which can be chosen from a class of 10 students.

Solution The number of such groups is $C(10,3)=\dfrac{10!}{3!\cdot(10-3)!}=\dfrac{10\cdot 9\cdot 8}{3\cdot 2\cdot 1}=120.$

EXAMPLE 18 There are 8 fruit pieces of different kinds including an apple on a tray. How many selections of 4 pieces of fruit can we make if we have to include the apple?

Solution If we have to include the apple, we need to select three pieces of fruit from the seven remaining: $C(7, 3) = 35$.

EXAMPLE 19 There are 10 players in a list. A basketball coach will choose 6 players from the list for a school team and make one of them the captain. In how many ways can the coach form the team?

Solution The coach can choose 6 players in 210 ways $\left(C(10,6)=\dfrac{10!}{6!\cdot(10-6)!}=210\right)$.

Additionally, any one of these six chosen players can be the captain. By the multiplication property, the coach can form the team in $C(10, 6)\cdot 6 = 1260$ ways.

EXAMPLE 20 In a group of 9 children, 4 children will be given apples, another 3 children will be given oranges and the rest will be given peaches. In how many ways can these fruits be given?

Solution We can choose four children from nine in $\binom{9}{4}$ ways and from the remaining five children we can choose three in $\binom{5}{3}$ ways. There will only be one way to choose the other two children.

So the total number of possible groupings is $\binom{9}{4}\cdot\binom{5}{3}\cdot 1 = 1260$.

Note that we can also solve this problem by treating it as a permutation with some identical elements.

EXAMPLE 21 A cafe offers chocolate, lemon, sour cherry and vanilla flavors of ice cream. A customer can choose one, two or three scoops but the flavours must all be different. How many different possible ice creams can a customer order?

Solution There are four types of ice cream.

The number of possible ice creams is $\binom{4}{1}+\binom{4}{2}+\binom{4}{3}=4+6+4=14.$

Notice that

$\binom{4}{1}$ and $\binom{4}{3}$ are equal: $\binom{4}{1}=\frac{4!}{1!\cdot 3!}=\binom{4}{3}.$

EXAMPLE 22 Classes 10A and 10B have 12 and 18 students respectively. A basketball team of 5 players will be formed by choosing 2 students from 10A and 3 students from 10B. How many different teams can be formed?

Solution The basketball team has five players.

We can choose 2 students from 12 students in $\binom{12}{2}$ ways.

We can choose 3 students from 18 students in $\binom{18}{3}$ ways.

So the team can be formed in $\binom{12}{2}\cdot\binom{18}{3}=66\cdot 816=53856$ ways.

EXAMPLE 23 How many three-digit numbers abc can we write which satisfy the condition $c < b < a$?

Solution Notice that the digits a, b and c must all be different. So any three-element set of digits $\{a, b, c\}$ will be enough to form a valid number, because we can just arrange the digits to satisfy the condition. For example, the digits set is $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and from the chosen subset $\{3, 5, 8\}$ we can form the number 853. So we just need to find the total number of three-digit subsets of the set of digits: $C(10, 3) = 120$ different numbers can be formed.

EXAMPLE 24 A watchmaker has 7 different jewels. He wants to choose four of them to decorate the quarters (3, 6, 9, 12) on the face of a clock. How many different decorations are possible?

Solution The watchmaker can choose four jewels in $\binom{7}{4}$ different ways and set them around the quarters on the dial in 4! different ways. So the total number of possible decorations is $\binom{7}{4}\cdot 4! = 840$.

(Notice that we cannot use circular permutation in this problem. Can you see why?)

EXAMPLE 25 Seven points are given as shown in the adjacent figure.

a. How many lines can be draw which pass through at least two of the points?

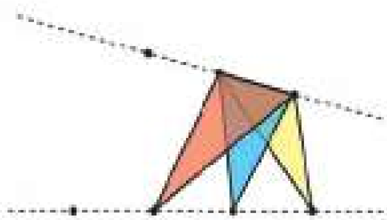
b. How many triangles can be formed using the points as vertices?

Solution a. Two lines are already given. There are three collinear points on the top line and four collinear points on the bottom line. Other lines can pass through one of the top and one of the bottom points. There are $3 \cdot 4 = 12$ such lines. Including the top and bottom line, there are $12 + 2 = 14$ possible lines.

b. For any triangle we want to draw, there are two cases:

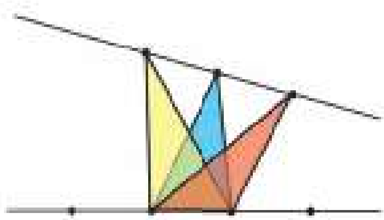
Case 1: A side is on the upper row and the vertex is a point on the lower row. Since two points determine a side, there can be $\binom{3}{2}\cdot 4 = 12$ such triangles.

Some of them are shown in the figure.



Case 2: A side is on the lower row and the vertex is on the upper row. There are $\binom{4}{2}\cdot 3 = 18$ such triangles.

Some of them are shown in the figure.



In conclusion, we can form $12 + 18 = 30$ triangles.

EXERCISES

A. Combinations of *r* Elements Selected from *n* Elements

1. Evaluate the expressions.

   a. $C(4, 2)$

   b. $C(6, 2) + C(8, 3)$

   c. $\dfrac{P(7,4)}{C(7,4)}$

   d. $\binom{9}{0}+\binom{9}{1}+\binom{9}{2}+\binom{9}{3}+\ldots+\binom{9}{9}$

   e. $\binom{13}{0}+\binom{13}{2}+\binom{13}{4}+\ldots+\binom{13}{12}$

2. Simplify the expressions.

   a. $2C(4, 2)$ b. $\dfrac{C(n, 3)}{P(n, 2)}$ c. $C(n, 2) + C(n, n-2)$

3. Solve the equations.

   a. $C(n, 2) = 15$ b. $\binom{n}{2}+20=\binom{8}{3}$

   c. $\binom{n+1}{n-1}=\binom{n-1}{2}+17$

4. List all the three-element subsets of the set $G = \{k, l, m, n, r\}$.

5. How many subsets of at least 4 elements does the set K = {★, ●, ☆, □, ○, ▶, ⬟} have?

6. How many of the three-element subsets of the set $H = \{a, b, c, d, e, f\}$ include the letter e?

7. In how many different ways can one novel, one biography and one poetry book be chosen from 3 novels, 4 biographies and 5 poetry books?

8. Snow White wants to choose 3 of the 7 dwarfs to clean her house. How many different groups can she choose?

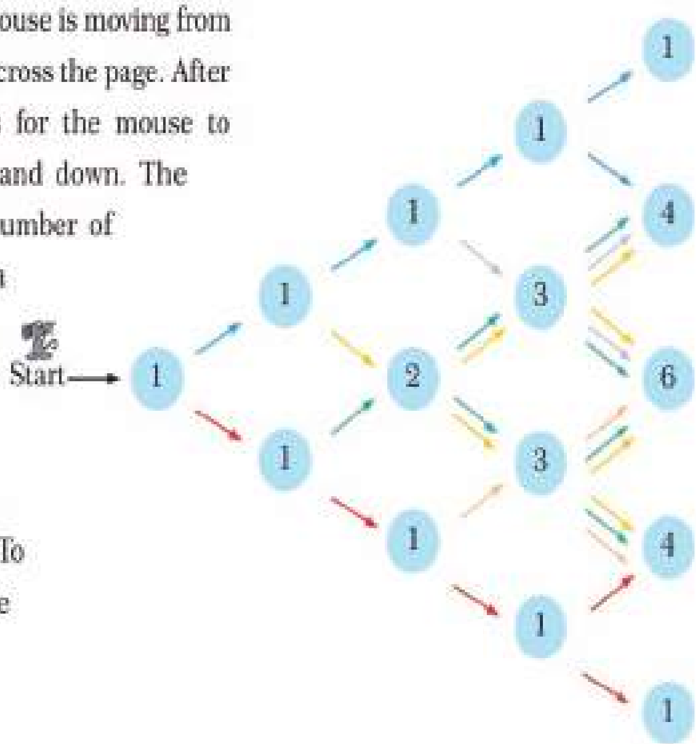
9. A computer programmer wants to set a key combination for an operation in a program. For this purpose, he will use two of the keys Shift, Ctrl or Alt together with one of 26 letters. How many different key combinations can he choose from?

10. In a group of 10 people, everybody shakes hands with everybody else. How many handshakes are there?

BINOMIAL EXPANSION

A. PASCAL'S TRIANGLE AND BINOMIAL EXPANSION

Look at the picture opposite. A mouse is moving from circle to circle from left to right across the page. After each circle there are two ways for the mouse to proceed: right and up or right and down. The number in each circle is the number of ways in which the mouse can reach that circle.



The numbers in the circles show a pattern which is known as **Pascal's triangle**. This triangle has many interesting properties. To understand them, let us move the triangle to an upright position.

Row																		Sum
0									1									1
1								1		1								2
2							1		2		1							4
3						1		3		3		1						8
4					1		4		6		4		1					16
5				1		5		10		10		5		1				32
6			1		6		15		20		15		6		1			64
7		1		7		21		35		35		21		7		1		128
8	1		8		28		56		70		56		28		8		1	256
	↓																↓	

First notice that each row begins and ends with 1.

Secondly, notice that the sum of any two consecutive terms in a row gives us the term between them on the next row. For instance, the number 15 marked in red in the sixth row is the sum of the 10 and 5 located above it. We can extend the triangle infinitely downwards by using this rule.

Notice also that the first row is row zero. For convenience, when we count the positions of the numbers in each row we also begin with zero (not 1). For example, number 21 marked in green is the second entry (not the third entry) in the seventh row. The entries in Pascal's triangle are related to the coefficients of the expansion of a binomial with a non-negative integer power. To understand the relationship, look at some binomial expansions with the first few powers and notice their coefficients.

$$(a+b)^0 = 1$$

$$(a+b)^1 = a+b$$
$$= 1a + 1b$$

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$= 1a^2 + 2ab + 1b^2$$

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$
$$= 1a^3 - 3a^2b + 3ab^2 - 1b^3$$

$$(x+2y)^4 = x^4 + 8x^3y + 24x^2y^2 + 32xy^3 + 16y^4$$
$$= 1x^4 + 4x^3(2y) + 6x^2(2y)^2 + 4x(2y)^3 + 1(2y)^4$$

We can see that the coefficients in each expansion are the same as the entries in the corresponding row of Pascal's triangle.

Remark

The expansion of a binomial expression to the n^{th} power has the following properties:

- There are $n + 1$ terms in the expansion.
- The coefficients of the terms in the expansion correspond to the entries in the n^{th} row of Pascal's triangle.
- The power of the first term in the binomial expression begins at n in the expansion and decreases by 1 in each term down to zero.
- The power of the second term in the binomial expression begins at zero in the expansion and increases by 1 in each term up to n.
- In the expansion of $(x + y)^n$, the sum of the exponents of x and y in each term is n.
- The sum of the coefficients of an expansion can be found by substituting 1 for each variable in the binomial expression.
- If the binomial expression is a polynomial then substituting zero for each variable in the binomial expression gives us the constant term of the expansion.

A constant term in an expression is a term that does not change with the variable.

EXAMPLE 26 How many terms are there in the expansion of $(x + y)^{12}$?

Solution Since the power is 12 ($n = 12$) there will be $n + 1 = 13$ terms.

EXAMPLE 27 Expand $(2x + y)^6$.

Solution The primary coefficients of the terms in the expansion will be the entries in the sixth row of Pascal's triangle: 1, 6, 15, 20, 15, 6, 1.

The first term of the binomial is $2x$. To avoid any mistakes, let us keep $2x$ in parantheses to begin with:

$$\begin{aligned}(2x + y)^6 &= (2x)^6 + 6(2x)^5y + 15(2x)^4y^2 + 20(2x)^3y^3 + 15(2x)^2y^4 + 6\cdot 2xy^5 + y^6\\ &= 64x^6 + 6(32x^5)y + 15(16x^4)y^2 + 20(8x^3)y^3 + 15\cdot 4x^2y^4 + 6\cdot 2xy^5 + y^6\\ &= 64x^6 + 192x^5y + 240x^4y^2 + 160x^3y^3 + 60x^2y^4 + 12xy^5 + y^6.\end{aligned}$$

EXAMPLE 28 Expand $(x^2 - 3y)^4$.

Solution We will use the entries in the fourth row as the coefficients: 1, 4, 6, 4, 1. If we write consider $(x^2 - 3y)^4$ as $(x^2 + (-3y))^4$ then

$$\begin{aligned}(x^2 - 3y)^4 &= (x^2)^4 - 4(x^2)^3(3y) + 6(x^2)^2(3y)^2 - 4(x^2)(3y)^3 + (3y)^4\\ &= x^8 - 12x^6y + 6x^4 9y^2 - 4x^2 27y^3 + 81y^4\\ &= x^8 - 12x^6y + 54x^4y^2 - 108x^2y^3 + 81y^4.\end{aligned}$$

EXAMPLE 29 Expand $\left(n+\dfrac{1}{n}\right)^7$.

Solution Use the seventh row:

$$\begin{aligned}\left(n+\frac{1}{n}\right)^7 &= n^7 + 7n^6\frac{1}{n} + 21n^5\left(\frac{1}{n}\right)^2 + 35n^4\left(\frac{1}{n}\right)^3 + 35n^3\left(\frac{1}{n}\right)^4 + 21n^2\left(\frac{1}{n}\right)^5 + 7n\left(\frac{1}{n}\right)^6 + \left(\frac{1}{n}\right)^7\\ &= n^7 + 7n^6\frac{1}{n} + 21n^5\frac{1}{n^2} + 35n^4\frac{1}{n^3} + 35n^3\frac{1}{n^4} + 21n^2\frac{1}{n^5} + 7n\frac{1}{n^6} + \frac{1}{n^7}\\ &= n^7 + 7n^5 + 21n^3 + 35n + \frac{35}{n} + \frac{21}{n^3} + \frac{7}{n^5} + \frac{1}{n^7}.\end{aligned}$$

EXAMPLE 30 Find the sum of the coefficients in the expansion of $(3x + y^2)^6$.

Solution If we substitute 1 for each variable in $(3x + y^2)^6$ we get $(3\cdot 1 + 1^2)^6 = 4^6 = 4096$.

EXAMPLE 31 Find the constant term in the expansion of $(7x + 3)^5$.

Solution Substitute zero for each variable in $(7x + 3)^5$: $(7 \cdot 0 + 3)^5 = 3^5 = 243$.

Check Yourself

Expand the binomials.

1. $(3x + y)^4$ 2. $\left(x - \frac{1}{x}\right)^3$ 3. $(2 - \sqrt{2})^5$

Answers

1. $81x^4 + 108x^3y + 54x^2y^2 + 12xy^3 + y^4$ 2. $x^3 - 3x + \frac{3}{x} - \frac{1}{x^3}$ 3. $232 - 164\sqrt{2}$

B. FINDING BINOMIAL TERMS USING COMBINATION

A laboratory mouse has been exposed to five types of virus. A scientist wishes to find out how many viruses are now present in the mouse. In how many ways could the mouse have been infected?

Infected with no viruses: $C(5,0) = 1$ (mouse is clean)

Infected with 1 type of virus: $C(5,1) = 5$

Infected with 2 types of virus: $C(5,2) = 10$

Infected with 3 types of virus: $C(5,3) = 10$

Infected with 4 types of virus: $C(5,4) = 5$

Infected with 5 types of virus: $C(5,5) = 1$.

Can you see the similarity between the number of combinations and the entries in the fifth row of Pascal's triangle?

Perhaps one of the most interesting characteristics of Pascal's triangle is its relationship with combination. We can describe this relationship simply: entry number r in row n is the number of subsets of r elements which can be taken from a set with n elements.

This gives us another interesting characteristic of Pascal's triangle: the sum of the terms in the n^{th} row of the triangle is 2^n (can you see why?).

The symmetrical property of Pascal's triangle can also be related to the combination rule

$$\binom{n}{r} = \binom{n}{n-r} \quad (n,\ r \in \mathbb{N} \text{ and } 0 \le r \le n).$$

For example, the third entry in the eighth row of the triangle is the same as the fifth entry in the same row since C(8, 3) = C(8, 5) = 56.

Let us now redraw Pascal's triangle using combination:

Row													
0							$\binom{0}{0}$						
1						$\binom{1}{0}$		$\binom{1}{1}$					
2					$\binom{2}{0}$		$\binom{2}{1}$		$\binom{2}{2}$				
3				$\binom{3}{0}$		$\binom{3}{1}$		$\binom{3}{2}$		$\binom{3}{3}$			
4			$\binom{4}{0}$		$\binom{4}{1}$		$\binom{4}{2}$		$\binom{4}{3}$		$\binom{4}{4}$		
5		$\binom{5}{0}$		$\binom{5}{1}$		$\binom{5}{2}$		$\binom{5}{3}$		$\binom{5}{4}$		$\binom{5}{5}$	
6	$\binom{6}{0}$		$\binom{6}{1}$		$\binom{6}{2}$		$\binom{6}{3}$		$\binom{6}{4}$		$\binom{6}{5}$		$\binom{6}{6}$
	↙												↘

Using the above triangle, we can generalize the expansion of a binomial to any power n as

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n.$$

Notice that the coefficient 1 in the first and last terms of the expansion is obtained from $\binom{n}{0}$ and $\binom{n}{n}$ respectively.

EXAMPLE 32 Expand $(x + y)^6$ using combination.

Solution $(x+y)^6 = x^6 + \binom{6}{1}x^{6-1}y + \binom{6}{2}x^{6-2}y^2 + \binom{6}{3}x^{6-3}y^3 + \binom{6}{4}x^{6-4}y^4 + \binom{6}{5}x^{6-5}y^5 + y^6$

$= x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

EXAMPLE 33 Expand $(2a-b)^5$.

Solution
$$(2a-b)^5=(2a)^5+\binom{5}{1}(2a)^{5-1}(-b)+\binom{5}{2}(2a)^{5-2}(-b)^2+\binom{5}{3}(2a)^{5-3}(-b)^3+\binom{5}{4}(2a)^{5-4}(-b)^4+(-b)^5$$
$$=32a^5-5\cdot16a^4b+10\cdot8a^3b^2-10\cdot4a^2b^3+5\cdot2ab^4-b^5$$
$$=32a^5-80a^4b+80a^3b^2-40a^2b^3+10ab^4-b^5$$

EXAMPLE 34 Expand $\left(\sqrt{x}+\frac{1}{x}\right)^7$.

Solution
$$\left(\sqrt{x}+\frac{1}{x}\right)^7=(\sqrt{x})^7+\binom{7}{1}(\sqrt{x})^6\left(\frac{1}{x}\right)+\binom{7}{2}(\sqrt{x})^5\left(\frac{1}{x}\right)^2+\binom{7}{3}(\sqrt{x})^4\left(\frac{1}{x}\right)^3+\binom{7}{4}(\sqrt{x})^3\left(\frac{1}{x}\right)^4$$
$$+\binom{7}{5}(\sqrt{x})^2\left(\frac{1}{x}\right)^5+\binom{7}{6}(\sqrt{x})\left(\frac{1}{x}\right)^6+\left(\frac{1}{x}\right)^7$$
$$=\sqrt{x^7}+7\sqrt{x^6}\frac{1}{x}+21\sqrt{x^5}\frac{1}{x^2}+35\sqrt{x^4}\frac{1}{x^3}+35\sqrt{x^3}\frac{1}{x^4}+21\sqrt{x^2}\frac{1}{x^5}+7\sqrt{x}\frac{1}{x^6}+\frac{1}{x^7}$$
$$=x^3\sqrt{x}+7x^2+21\sqrt{x}+\frac{35}{x}+\frac{35\sqrt{x}}{x^3}+\frac{21}{x^4}+\frac{7\sqrt{x}}{x^6}+\frac{1}{x^7}$$

The relation between combination and Pascal's triangle helps us to calculate any particular term in a binomial expansion without writing out the entire expansion.

For example, suppose that we are asked to find the third term in the expansion of $(x-2y)^3$. Using our knowledge of the properties of binomial expansion, we can say that $2y$ will have exponent 2 in this term and x will have exponent 1 since the sum of the exponents must be 3. Now we only need to find the coefficient, which we can calculate as $\binom{3}{2}=3$.

So the third term is $\binom{3}{2}x(2y)^2=3x\cdot4y^2=12xy^2$. We can easily check this against the full expansion: $(x-2y)^3=x^3-6x^2y+\underline{12xy^2}-8y^3$.

We can formulize our findings as follows:

Remark

The r^{th} entry in the expansion of $(x+y)^n$ is $\binom{n}{r-1}x^{n-(r-1)}y^{r-1} = \binom{n}{r-1}x^{n-r+1}y^{r-1}$.

EXAMPLE 35 Find the ninth term in the expansion of $(a+b)^{12}$.

Solution Substitute $n = 12$ and $r = 9$ in the formula: $\binom{12}{9-1}x^{12-9+1}y^{9-1} = 495a^4b^8$.

EXAMPLE 36 Find the sixth term in the expansion of $(2x+y)^9$.

Solution We will use $n = 9$ and $r = 6$. So $r - 1 = 5$ and the sixth term is

$\binom{9}{5}(2x)^{9-5}y^5 = 126 \cdot 16x^4y^5 = 2016x^4y^5$.

EXAMPLE 37 What is the coefficient of the fourth term in the expansion of $(2x-4y)^7$?

Solution $r = 4$ means $r - 1 = 3$. So the fourth term is $\binom{7}{3}(2x)^{7-3}(-4y)^3 = 35 \cdot 16x^4(-64)y^3$. From this we can calculate the coefficient to be -35840.

PROBABILITY

Definition

experiment, outcome, sample space, event, simple event

An **experiment** is an activity or a process which has observable results. For example, rolling a die is an experiment.

The possible results of an experiment are called **outcomes**. The outcomes of rolling a die once are 1, 2, 3, 4, 5, or 6.

The set of all possible outcomes of an experiment is called the **sample space** for the experiment. The sample space for rolling a die once is {1, 2, 3, 4, 5, 6}.

An **event** is a subset of (or a part of) a sample space. For example, the event of an odd number being rolled on a die is {1, 3, 5}.

If the sample space of an experiment with n outcomes is $S = \{e_1, e_2, e_3, e_4, \ldots, e_n\}$ then the events $\{e_1\}, \{e_2\}, \{e_3\}, \ldots, \{e_n\}$ which consist of exactly one outcome are called **simple events**.

EXAMPLE 38 What is the sample space for the experiment of tossing a coin?

Solution There are two possible outcomes: tossing heads and tossing tails. So the sample space is {heads, tails}, or simply $\{H, T\}$.

EXAMPLE 39 Write the sample space for tossing a coin three times.

Solution The sample space is $\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$.

EXAMPLE 40 The sample space for an experiment is {1, 2, 3, 4, 5, 6, 7, 8, 9}. Write the event that the result is a prime number.

Solution The event is {2, 3, 5, 7}.

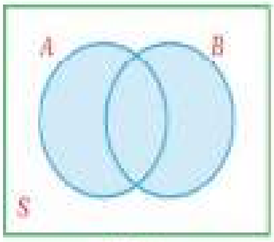
Definition

union and intersection of events, complement of an event

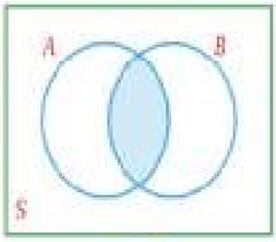
The **union** of two events A and B is the set of all outcomes which are in A and/or B. It is denoted by $A \cup B$.

The **intersection** of two events A and B is the set of all outcomes in both A and B. It is denoted by $A \cap B$.

The **complement** of an event A is the set of all outcomes in the sample space that are not in the event A. It is denoted by A' (or A^C).



$A \cup B$
(union of A and B)



$A \cap B$
(intersection of A and B)

A'
(complement of A)

EXAMPLE 41 Consider the events $A = \{1, 2, 3, 4\}$ and $B = \{4, 5, 6\}$ in the experiment of rolling a die. Write the events $A \cup B$, $A \cap B$ and A'.

Solution The sample space for this experiment is $\{1, 2, 3, 4, 5, 6\}$. Therefore,

$A \cup B = \{1, 2, 3, 4, 5, 6\}$ (the set of all outcomes in events A and/or B);

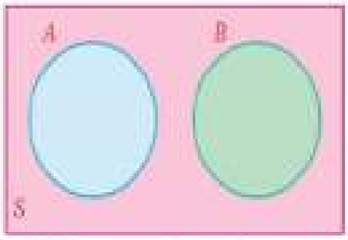
$A \cap B = \{4\}$ (the set of all common outcomes in A and B);

$A' = \{5, 6\}$ (the set of all outcomes in the sample space that are not in event A).

Definition

mutually exclusive events

Two events which cannot occur at the same time are called **mutually exclusive events**. In other words, if two events have no outcome in common then they are mutually exclusive events.



A and B are mutually exclusive events.

For example, consider the sample space for rolling a die. The event that the number rolled is even and the event that the number rolled is odd are two mutually exclusive events, since $E = \{2, 4, 6\}$ and $O = \{1, 3, 5\}$ have no outcome in common.

Now we are ready to define the concept of probability of an event.

Definition probability of an event

Let E be an event in a sample space S in which all the outcomes are equally likely to occur. Then the **probability of event** E is $P(E) = \frac{n(E)}{n(S)}$, where $n(E)$ is the number of outcomes in event E and $n(S)$ is the number of outcomes in the sample space S.

EXAMPLE 42 A coin is tossed. What is the probability of obtaining a tail?

Solution The sample space for this experiment is $\{H, T\}$ and the event is $\{T\}$, so $n(S) = 2$ and $n(E) = 1$.

So the desired probability is $P(E) = \frac{n(E)}{n(S)} = \frac{1}{2}$.

EXAMPLE 43 I roll a die. What is the probability that the number rolled is odd?

Solution The sample space is $S = \{1, 2, 3, 4, 5, 6\}$ and the event that the number is odd is $E = \{1, 3, 5\}$.

So the probability is $P(E) = \frac{n(E)}{n(S)} = \frac{3}{6} = \frac{1}{2}$.

EXAMPLE 44 A coin is tossed three times. What is the probability of getting only one head?

Solution The sample space is

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$ and the desired event is $E = \{HTT, THT, TTH\}$. So the probability is $P(E) = \frac{3}{8}$.

EXAMPLE 45 The integers 1 through 15 are written on separate cards. You are asked to pick a card at random. What is the probability that you pick a prime number?

Solution There are fifteen numbers in the sample space. The primes in the set are 2, 3, 5, 7, 11 and 13. So the desired probability is $\frac{6}{15}=\frac{2}{5}$.

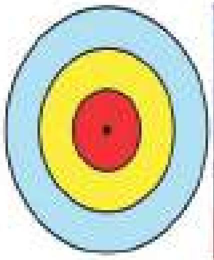
Remark

Since the number of outcomes in an event is always less than or equal to the number of outcomes in the sample space, $\frac{n(E)}{n(S)}$ is always less than or equal to 1.

Also, the smallest possible number of outcomes in an event is zero. So the smallest possible probability ratio is $\frac{n(E)}{n(S)}=\frac{0}{n(S)}=0$.

In conclusion, the probability of an event always lies between 0 and 1, i.e. $0 \le P(E) \le 1$.

EXAMPLE 46 A child is throwing darts at the board shown in the figure. The radii of the circles on the board are 3 cm, 6 cm and 9 cm respectively. What is the probability that the child's dart lands in the red circle, given that it hits the board?



Solution We know from geometry that the area of a circle with radius r is πr^2. Hence the area of the red circle is $\pi 3^2 = 9\pi$ cm^2 and the area of the pentire board is $\pi 9^2 = 81\pi$ cm^2.

We can consider the area of each region as the number of outcomes in the related event.

So the probability that the dart lands in the red circle is $P(\text{red})=\frac{n(\text{red})}{n(\text{board})}=\frac{9\pi}{81\pi}=\frac{1}{9}$.

As the probability of an event gets closer to 1, the event is more likely to occur. As it gets closer to zero, the event is less likely to occur. In the previous example, the probability is close to zero so the event is not very likely. However, note that $\frac{1}{9}$ does not tell us anything about what will actually happen as the child is throwing the darts. The child will not necessarily hit the red circle once every nine darts. He might hit it three times with nine darts, or not at all. But if the child played for a long time and we looked at the ratio of the red hits, to the other hits we would find that it is close to $\frac{1}{9}$.

Definition certain event, impossible event

An event whose probability is 1 is called a **certain event**. An event whose probability is zero is called an **impossible event**.

EXAMPLE 47 A student rolls a die. What is the probability of each event?

a. the number rolled is less than 8

b. the number rolled is 9

Solution The sample space is $S = \{1, 2, 3, 4, 5, 6\}$.

a. We can see that every number in the sample space is less than 8.

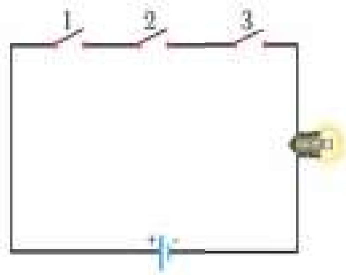
So the event is $E = \{1, 2, 3, 4, 5, 6\}$.

Therefore the probability that the number is less than 8 is

$P(E) = \frac{6}{6} = 1$, which means the event is a certain event.

b. Since it is not possible to roll a 9 with a single die, the event is an empty set ($E = \varnothing$). So the probability is $P(E) = \frac{0}{6} = 0$, which means the event is an impossible event.

EXAMPLE 48 A small child randomly presses all the switches in the circuit shown opposite. What is the probability that the bulb lights?



Solution Each switch can be either open or closed. Let us write O to mean an open switch and C to mean a closed switch. Then the sample space contains $2 \cdot 2 \cdot 2 = 8$ outcomes, namely $\{O_1O_2O_3, O_1O_2C_3, O_1C_2C_3, O_1C_2O_3, C_1O_2O_3, C_1O_2C_3, C_1C_2O_3, C_1C_2C_3\}$.

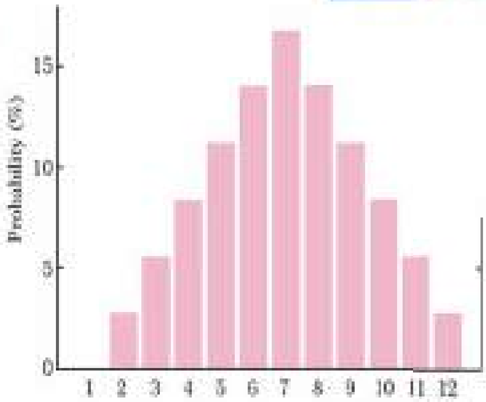
The bulb only lights when all the switches are closed. So the desired probability is $\frac{1}{8}$.

EXAMPLE 49 In a game, a player bets on a number from 2 to 12 and rolls two dice. If the sum of the spots on the dice is the number he guessed, he wins the game. Which number would you advise the player to bet on? Why?

Solution There is no difference between rolling a die twice and rolling two dice together. Let us make a table of the possible outcomes of rolling the dice:

	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

→ 7



We can see that there are six ways of rolling 7 with two dice. This is the most frequent outcome of the game, so the player should bet on 7. As there are $6 \cdot 6 = 36$ outcomes in the sample space, the probability of rolling 7 is $\frac{6}{36} = \frac{1}{6}$, which is the highest probability in the game.

Check Yourself

1. A family with three children is selected from a population and the genders (male or female) of the children are written in order, from oldest to youngest. If M represents a male child and F represents a female child, write the sample space for this experiment.
2. A student rolls a die which has one white face, two red faces and three blue faces. What is the probability that the top face is blue?
3. Two dice are rolled together. What is the probability of obtaining a sum less than 6?
4. A box contains 15 light bulbs, 4 of which are defective. A bulb is selected at random. What is the probability that it is not defective?
5. Three dice are rolled together. What is the probability of rolling a sum of 15?

Answers

1. $\{MMM, MMF, MFM, FMM, MFF, FMF, FFM, FFF\}$ 2. $\frac{1}{2}$ 3. $\frac{5}{18}$ 4. $\frac{11}{15}$ 5. $\frac{5}{108}$

In this section we will learn some rules of probability which are frequently used for solving problems.

Rule

rules of probability

1. For every event E, $0 \leq P(E) \leq 1$.
2. For a sample space S, $P(S) = 1$.
3. For two mutually exclusive events A and B, $P(A \cup B) = P(A) + P(B)$.
4. For any two events A and B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
5. For any event A, $P(A) + P(A') = 1$. In other words, $P(A') = 1 - P(A)$.

Note

Many problems in probability are written in natural language. The key word for recognizing the union operation ($\cup$) in a written problem is 'or'. When we use the word 'or' (A or B) in mathematics, we mean A or B or both.

The key word for recognizing the intersection operation ($\cap$) in a written problem is 'and'. When we use the word 'and' (A and B) in mathematics, we mean both A and B.

EXAMPLE 50 A die is rolled. Find the probability that it shows 3 or 5.

Solution Let T mean the die shows 3 and F mean the die shows 5. Then '3 or 5' means $T \cup F$.

Since T and F are mutually exclusive events, by the rules of probability we can write

$$P(T \cup F) = P(T) + P(F)$$

$$= \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.$$ So the probability is $\frac{1}{3}$.

EXAMPLE 51 A die is rolled. Find the probability that it shows an even number or a prime number.

Solution The possible prime numbers are 2, 3 and 5 and the even numbers are 2, 4 and 6. Showing an even number (E) or a prime number (P) are not mutually exclusive events, since the outcome is in both events.

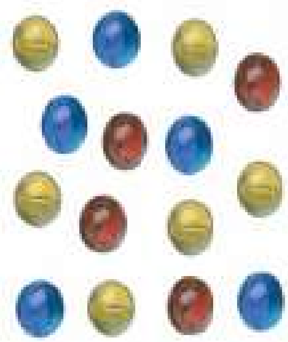
Since $P(2) = \frac{1}{6}$, $P(E \cap P) = \frac{1}{6}$.

So the probability of E or P is $P(E \cup P) = P(E) + P(P) - P(E \cap P)$

$$= \frac{1}{2} + \frac{1}{2} - \frac{1}{6}$$

$$= \frac{5}{6}.$$

EXAMPLE 52 An urn contains five blue marbles, four red marbles and six yellow marbles. We want to take one marble from the urn. What is the probability of taking a red or a yellow marble?



Solution 1 Since a marble cannot be both red and yellow, drawing a red marble and drawing a yellow marble are mutually exclusive events. So the probability is

$$P(R \cup Y) = P(R) + P(Y) = \frac{4}{15} + \frac{6}{15} = \frac{10}{15} = \frac{2}{3}.$$

Solution 2 We can also solve the problem in another way. Let E be the event that a red or yellow marble is drawn. Then the complement of E (written E') is the event that neither a red nor a yellow marble is drawn. In other words, E' is the event that a blue marble is drawn.

We also know that $P(E) + \underbrace{P(E')}_{\text{drawing a blue marble}} = 1.$

So the probability of drawing a red or yellow marble is $P(E) = 1 - P(E') = 1 - \frac{5}{15} = \frac{10}{15} = \frac{2}{3}$

EXAMPLE 53 We have twenty cards numbered from 1 to 20. A card is drawn at random. What is the probability of drawing an even number or a number divisible by 3?

Solution Let the event that an even number is drawn be $E = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$ and the event that a number divisible by 3 is drawn be $T = \{3, 6, 9, 12, 15, 18\}$. We can see that $E \cap T = \{6, 12, 18\}$.

So we can write $P(E \cup T) = P(E) + P(T) - P(E \cap T) = \frac{10}{20} + \frac{6}{20} - \frac{3}{20} = \frac{13}{20}.$

EXAMPLE 54 A coin is tossed four times. What is the probability that the coin shows tails at least once?

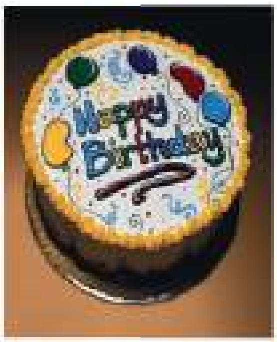
Solution The sample space contains $2^4 = 16$ outcomes. If E is the event that we get tails at least once then E' is the event that we get no tails. In other words, E' is the event that we get heads three times (can you see why?). Since there is only one way to do this, $P(E') = \frac{1}{16}$.

So $P(E) = 1 - P(E') = 1 - \frac{1}{16} = \frac{15}{16}.$

We can check this result with the sample space:

$S = \{HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTHT, THHT, TTHH, HTTH, THTH, HTTT, THTT, TTHT, TTTH, TTTT\}$.

EXAMPLE 55 A group of 6 people is selected at random. What is the probability that at least two of them have the same birthday?



Solution First let us assume that there are 365 days in a year. Then the sample space for one person's birthday has 365 outcomes because there are 365 possible dates for a contains. Let the desired event be A. Then A' is the event that none of these six people have a common birthday.

So A' contains $365 \cdot 364 \cdot 363 \cdot 362 \cdot 361 \cdot 360$ outcomes.

Let E be the sample space for the experiment. Then E contains 365^6 possible outcomes, because there are six people.

So the probability of A is $1-\dfrac{n(A')}{n(E)}=1-\dfrac{365 \cdot 364 \cdot 363 \cdot 362 \cdot 361 \cdot 360}{365^6} \cong 0.05.$

Check Yourself

1. A die is rolled. What is the probability that the die shows a number greater than 3 or an even number?
2. A number is drawn at random from the set $A = \{1, 2, 3, ... ,100\}$.
   a. What is the probability that the number is divisible by both 2 and 3?
   b. What is the probability that the number is divisible by 2 or 3?

Answers

1. $\dfrac{2}{3}$ 2. a. $\dfrac{4}{25}$ b. $\dfrac{67}{100}$

EXERCISES

1. An urn contains 5 red, 3 yellow and 6 white marbles. We draw a marble from the urn at random. What is the probability of drawing a red or a yellow marble?

2. An urn contains 3 black, 4 red and 2 blue marbles. What is the probability that a marble drawn at random is not blue?

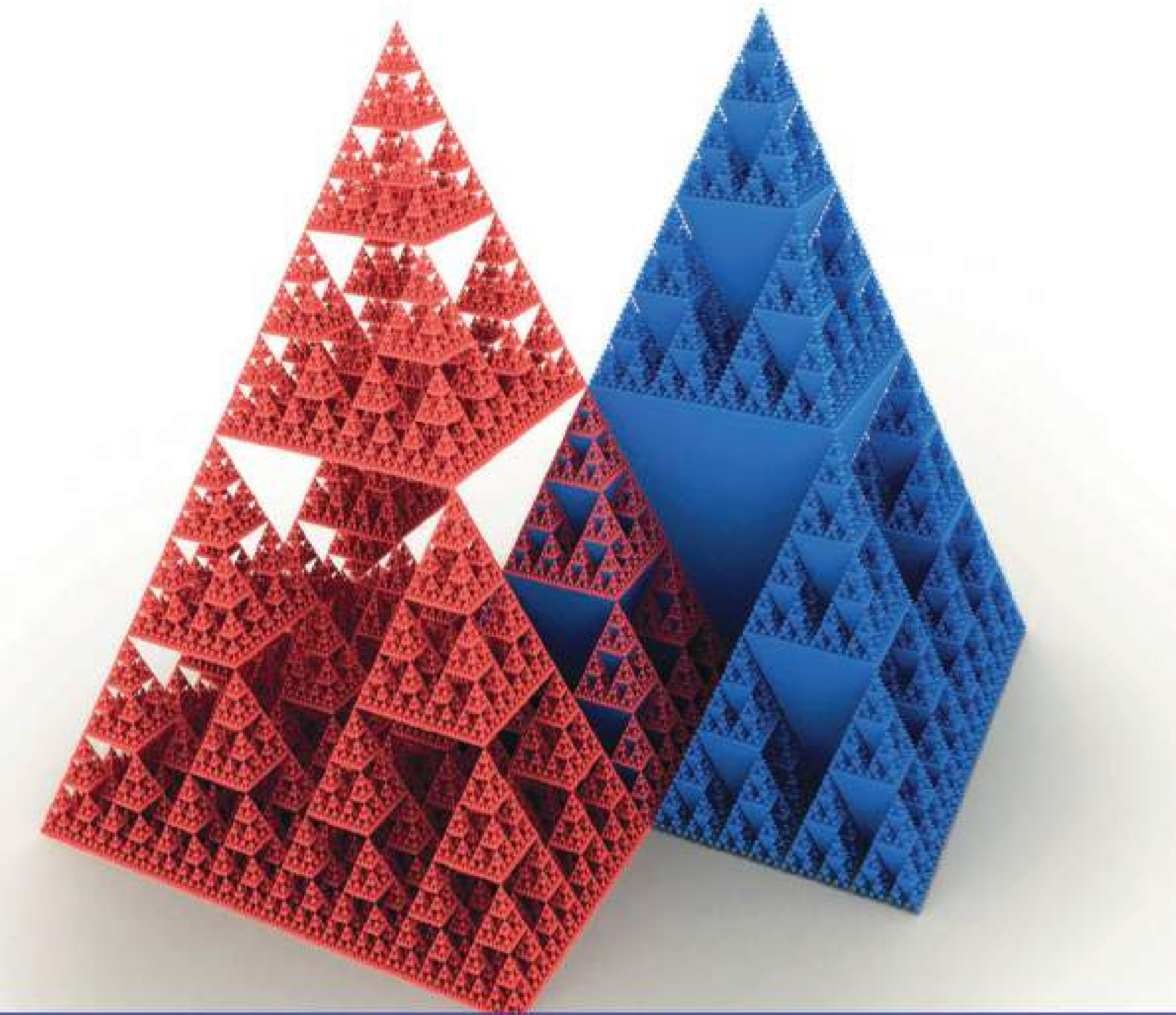
3. A fair die is rolled. What is the probability of rolling an even number or a prime number?

4. A fair die is rolled. What is the probability of rolling a number which is less than 5 or greater than 2?

5. You draw a card from a well-shuffled deck of 52 cards. What is the probability of drawing a king or a queen?

6. Two dice are rolled. What is the probability of obtaining a sum of 6 or 10?

7. A bag contains 4 red balls, 3 yellow balls, 5 green balls and 2 black balls. Find the probability that a ball drawn at random is neither black nor red.



Chapter 9

MATRICES

MATRICES

A. BASIC CONCEPTS

Three car dealers each sell three models of a car. The table below shows the number of cars each dealer sold in a given month.

	Dealer 1	Dealer 2	Dealer 3
Model A	5	7	4
Model B	3	4	9
Model C	2	0	1

We can organize data like this in a **matrix**. The matrix for our car dealer data is

$$A = \begin{array}{c} \\ \text{Model A} \\ \text{Model B} \\ \text{Model C} \end{array} \begin{array}{c} \begin{matrix} D_1 & D_2 & D_3 \end{matrix} \\ \begin{bmatrix} 5 & 7 & 4 \\ 3 & 4 & 9 \\ 2 & 0 & 1 \end{bmatrix} \end{array}.$$

The plural form of matrix is **matrices**, pronounced 'may-trih-sees'.

Two large square brackets contain the numbers in the matrix. This matrix has three rows and three columns.

The column $\begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$ represents the cars sold by the first dealer. The row $[5 \;\; 7 \;\; 4]$ represents all the model A cars sold by the three dealers.

Matrices give us an effective way of organizing and manipulating data in different problems. We can begin our study of matrices with a more formal definition.

Definition

matrix

A **matrix** is a rectangular arrangement of numbers in rows and columns.

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13}\cdots & a_{1n} \\ a_{21} & a_{22} & a_{23}\cdots & a_{2n} \\ a_{31} & a_{32} & a_{33}\cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3}\cdots & a_{mn} \end{bmatrix}}_{n \text{ columns}} \Big\} m \text{ rows}$$

The horizontal lines of numbers in a matrix are called **rows** and the vertical lines are called **columns**. The number of rows and the number of columns determine the **dimensions** (also called the **order**) of the matrix. A matrix with m rows and n columns has dimensions $m \times n$ and is called an $m \times n$ (read as 'm by n') **matrix**. Notice that the number of rows is always given first.

Each number in a matrix is called an **entry** of the matrix. a_{ij} means the entry in the ith row and jth column of the matrix A.

$m \times n$

rows × columns

3 columns

$$A = \begin{bmatrix} 2 & 5 & 4 \\ 6 & 3 & 0 \end{bmatrix}$$

2 rows

Matrix A is a 2×3 ('two by three') matrix.

a_{13}

1st row 3rd column

$$A = \begin{bmatrix} 2 & 5 & 4 \\ 6 & 3 & 0 \end{bmatrix}$$

a_{13} is the entry in the first row and the third column: $a_{13} = 4$.

EXAMPLE 1 Write the dimensions of each real matrix.

If each entry of a matrix is a real number then the matrix is called a **real matrix**.

a. $A = \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix}$ b. $B = \begin{bmatrix} \pi & -1 & \frac{1}{2} \end{bmatrix}$ c. $C = \begin{bmatrix} 0.5 \\ -2 \\ 7 \end{bmatrix}$ d. $D = \begin{bmatrix} 2 & 0 & 4 & 1 \\ 5 & 9 & 0 & 7 \\ 1 & 0 & 4 & 9 \end{bmatrix}$

Solution a. $[A]_{2\times2}$ is a 2×2 matrix b. $[B]_{1\times3}$ is a 1×3 matrix

c. $[C]_{3\times1}$ is a 3×1 matrix d. $[D]_{3\times4}$ is a 3×4 matrix

$[A]_{m\times n}$ means a matrix A with m rows and n columns.

EXAMPLE 2 $A = \begin{bmatrix} 2 & 0 & 4 & 1 \\ 5 & 9 & 0 & 7 \\ 1 & 0 & 4 & 9 \end{bmatrix}$ is given. Write each matrix entry.

a. a_{31} b. a_{23} c. a_{34} d. a_{22}

Solution a. 1 b. 0 c. 9 d. 9

EXAMPLE 3 Given $A = \begin{bmatrix} -3 & 0 & 5 \\ \sqrt{3} & 1 & \frac{3}{2} \end{bmatrix}$, find $2a_{13} - 3a_{21}^2 - 4a_{23}$.

Solution We have $a_{13} = 5$, $a_{21} = \sqrt{3}$ and $a_{23} = \frac{3}{2}$, so the expression becomes

$$(2\cdot5) - 3(\sqrt{3})^2 - (4\cdot\frac{3}{2}) = 10 - 9 - 6 = -5.$$

B. TYPES OF MATRICES

> A matrix with only one row, such as $\begin{bmatrix}1 & 3 & 2\end{bmatrix}$, is called a **row matrix**.
> Likewise, a **column matrix** such as $\begin{bmatrix}1\\3\\4\end{bmatrix}$ has only one column.

1. Square Matrix

A **square matrix** is a matrix which has the same number of rows and columns.

$[2]$, $\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ and $\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 6 & 0\end{bmatrix}$ are all square matrices. We say that a square matrix has **order n** if it has n rows and n columns.

2. Zero Matrix

A **zero matrix** is a matrix whose entries are all zeros. We write **0** to mean a zero matrix.

$[0]$, $\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ and $\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$ are all zero matrices.

> The **main diagonal** of a square matrix always runs from top left to bottom right.
>
> $\begin{bmatrix}a_{11} & \\ & a_{nn}\end{bmatrix}$
>
> *main diagonal*

3. Identity Matrix

A square matrix whose **main diagonal elements (from top left to bottom right)** are 1 and whose other entries are all zero is called an **identity matrix**. We write **I** to mean the identity matrix.

$[1]$, $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ and $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$ are all identity matrices. $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ is not an identity matrix.

4. Diagonal Matrix

A square matrix in which all the entries except the main diagonal entries are zero is called a **diagonal matrix**.

$\begin{bmatrix}-1 & 0\\0 & 8\end{bmatrix}$ and $\begin{bmatrix}2 & 0 & 0\\0 & 5 & 0\\0 & 0 & 0\end{bmatrix}$ are diagonal matrices.

5. Scalar Matrix

A square matrix whose main diagonal elements are all equal ($a_{11} = a_{22} = a_{33} = ...$) and whose other entries are all zero is called a **scalar matrix**.

$[8]$, $\begin{bmatrix}-5 & 0\\0 & -5\end{bmatrix}$ and $\begin{bmatrix}7 & 0 & 0\\0 & 7 & 0\\0 & 0 & 7\end{bmatrix}$ are all scalar matrices.

Notice that a scalar matrix is a type of diagonal matrix.

Answers

1. a. 3×1, column matrix b. 3×3, square matrix, diagonal matrix, scalar matrix
   c. 4×4, square matrix

2. $a_{14} = 8, a_{22} = 3, a_{34} = 1$

C. EQUAL MATRICES

Definition equal matrices

Two matrices are called **equal matrices** if they have the same dimension and their corresponding entries are all equal. We write $A = B$ to mean that two matrices A and B are equal.

Notice that $[1 \quad 2 \quad 3] \neq \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, because these matrices do not have the same dimension: $(1\times3) \neq (3\times1)$.

EXAMPLE 4 Find a_{11}, a_{12}, a_{21} and a_{22} in the matrix equation.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 0 \end{bmatrix}$$

Solution Since the two matrices are equal, their corresponding entries are equal, and so

$a_{11} = 2, \quad a_{12} = -1,$

$a_{21} = -3, \quad a_{22} = 0.$

EXAMPLE 5 $\begin{bmatrix} x+y & -3 \\ 3 & x-y \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 3 & 2x+3y \end{bmatrix}$ is given. Find x, y and z.

Solution Since the two matrices are equal, their corresponding entries are equal. So

$x + y = 2$ (1)

$x - y = 2x + 3y \Rightarrow x + 4y = 0.$ (2)

Solving (1) and (2) for x and y gives us $x = \frac{8}{3}, \ y = -\frac{2}{3}$.

D. OPERATIONS ON MATRICES

1. Matrix Addition

Although we can always add two real numbers together, we cannot always add two matrices. In fact, we can only perform matrix addition on matrices with equal dimensions. To add or subtract two matrices A and B, we simply add or subtract corresponding entries.

$$\underset{A}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} + \underset{B}{\begin{bmatrix} e & f \\ g & h \end{bmatrix}} = \begin{bmatrix} (a+e) & (b+f) \\ (c+g) & (d+h) \end{bmatrix}$$

EXAMPLE 6

$A=\begin{bmatrix} 5 & 0 & 4 \\ 2 & 3 & -1 \end{bmatrix}$, $B=\begin{bmatrix} 2 & 2 & 1 \\ 5 & 3 & 7 \end{bmatrix}$ and $C=\begin{bmatrix} 1 & 5 \\ 3 & 0 \end{bmatrix}$ are given. Write the matrices.

a. $A+B$ b. $A-B$ c. $A+C$

Solution

a. Since the matrices have the same dimensions, we can add them.

$$\begin{bmatrix} 5 & 0 & 4 \\ 2 & 3 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 2 & 1 \\ 5 & 3 & 7 \end{bmatrix} = \begin{bmatrix} 5+2 & 0+2 & 4+1 \\ 2+5 & 3+3 & -1+7 \end{bmatrix} = \begin{bmatrix} 7 & 2 & 5 \\ 7 & 6 & 6 \end{bmatrix}$$

b. Since the matrices have the same dimensions, we can subtract them.

$$\begin{bmatrix} 5 & 0 & 4 \\ 2 & 3 & -1 \end{bmatrix} - \begin{bmatrix} 2 & 2 & 1 \\ 5 & 3 & 7 \end{bmatrix} = \begin{bmatrix} 5-2 & 0-2 & 4-1 \\ 2-5 & 3-3 & -1-7 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 3 \\ -3 & 0 & -8 \end{bmatrix}$$

Notice that $A-B \neq B-A$.

c. $A+C$ is undefined, since A and B have different dimensions.

EXAMPLE 7

Find each matrix sum or difference.

a. $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} -1 & -2 \\ -4 & -3 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ c. $\begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

Solution

a. $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} -1 & -2 \\ -4 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 2 & -8 \end{bmatrix}$

2. Multiplying a Matrix and a Scalar

In matrix algebra, a real number is often called a *scalar*. To multiply a matrix by a scalar, we multiply each entry in the matrix by the scalar. This operation is called **scalar multiplication**.

$$c\cdot\begin{bmatrix}a & b\\ b & d\end{bmatrix}=\begin{bmatrix}c\cdot a & c\cdot b\\ c\cdot b & c\cdot d\end{bmatrix}$$

EXAMPLE 8 The matrices $A=\begin{bmatrix}1 & 2 & 4\\ -3 & 0 & -1\\ 2 & 1 & 2\end{bmatrix}$ and $B=\begin{bmatrix}2 & 0 & 0\\ 1 & -4 & 3\\ -1 & 3 & 2\end{bmatrix}$ are given. Perform the matrix operations.

a. $3A$ b. $-B$ c. $3A-B$

Solution

a. $3A=3\cdot\begin{bmatrix}1 & 2 & 4\\ -3 & 0 & -1\\ 2 & 1 & 2\end{bmatrix}=\begin{bmatrix}3\cdot 1 & 3\cdot 2 & 3\cdot 4\\ 3\cdot(-3) & 3\cdot 0 & 3\cdot(-1)\\ 3\cdot 2 & 3\cdot 1 & 3\cdot 2\end{bmatrix}=\begin{bmatrix}3 & 6 & 12\\ -9 & 0 & -3\\ 6 & 3 & 6\end{bmatrix}$

b. $-B=(-1)\begin{bmatrix}1 & -4 & 3\\ -1 & 3 & 2\end{bmatrix}=\begin{bmatrix}-1 & 4 & -3\\ 1 & -3 & -2\end{bmatrix}$

c. $3A-B=3A+(-B)=\underbrace{\begin{bmatrix} & & \\ -9 & 0 & -3\\ 6 & 3 & 6\end{bmatrix}}_{\text{from a}}+\underbrace{\begin{bmatrix}2 & 0 & 0\\ -1 & 4 & -3\\ 1 & -3 & -2\end{bmatrix}}_{\text{from b}}=\begin{bmatrix}1 & 6 & 12\\ -10 & 4 & -6\\ 7 & 0 & 4\end{bmatrix}$

EXAMPLE 9 Solve the matrix equation for x and y. $2\left(\begin{bmatrix}3 & 2x\\ 6 & 4\end{bmatrix}+\begin{bmatrix}-3 & 5\\ -3y & 3\end{bmatrix}\right)=\begin{bmatrix}0 & 18\\ 6 & 14\end{bmatrix}$

Solution Simplify the left side of the equation:

$$2\left(\begin{bmatrix}3 & 2x\\ 6 & 4\end{bmatrix}+\begin{bmatrix}-3 & 5\\ -3y & 3\end{bmatrix}\right)=\begin{bmatrix}0 & 18\\ 6 & 14\end{bmatrix}$$

$$2\begin{bmatrix}3-3 & 2x+5\\ 6-3y & 4+3\end{bmatrix}=\begin{bmatrix}0 & 18\\ 6 & 14\end{bmatrix}$$

$$\begin{bmatrix}0 & 4x+10\\ 12-6y & 14\end{bmatrix}=\begin{bmatrix}0 & 18\\ 6 & 14\end{bmatrix}.$$

Equate corresponding entries and solve the two resulting equations:

$$4x+10=18 \qquad 12-6y=6$$

$$4x=8 \qquad 6y=6$$

$$x=2 \qquad y=1.$$

Check Yourself

Given the matrices $A=\begin{bmatrix}6 & -1 & 4\\ 2 & 4 & 3\end{bmatrix}$ and $B=\begin{bmatrix}1 & 4 & 2\\ -1 & 5 & 0\end{bmatrix}$, find

a. $-A$. b. $2A-3B$. c. $5A+B$.

Answers

a. $\begin{bmatrix}-6 & 1 & -4\\ -2 & -4 & -3\end{bmatrix}$ b. $\begin{bmatrix}9 & -14 & 2\\ 7 & -7 & 6\end{bmatrix}$ c. $\begin{bmatrix}31 & -1 & 22\\ 9 & 25 & 15\end{bmatrix}$

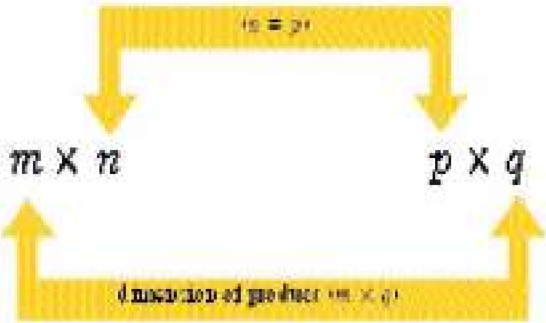
3. Matrix Multiplication

It is important to check the dimensions of two matrices before we start to multiply them. If matrix A has dimension $m \times n$ and matrix B has dimension $p \times q$, then the product AB only exists if $n = p$. Furthermore, the product will have dimension $m \times q$.

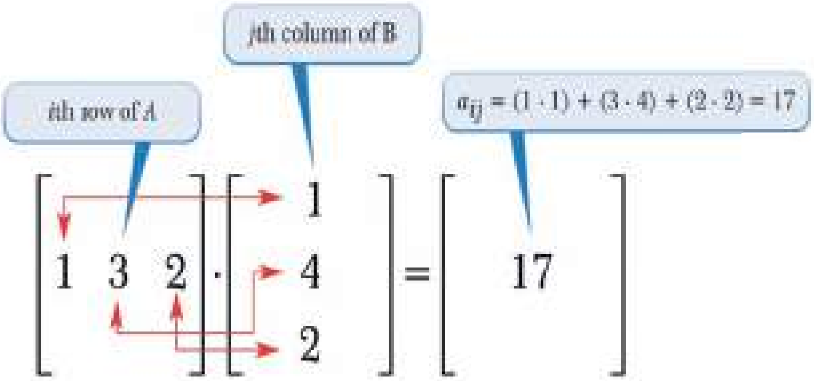
If A is an $m \times n$ matrix and B is an $n \times p$ matrix then the product AB is an $m \times p$ matrix.

$$\underset{m\times n}{A} \cdot \underset{n\times p}{B} = \underset{m\times p}{AB}$$

(equal; dimensions of AB)



We obtain each entry in the matrix AB (the product of A and B) from a row of A and a column of B as follows: multiply the entries in the ith row of A by the entries in the jth column of B and add the results to get a_{ij} in AB.



EXAMPLE 10 $A=\begin{bmatrix}2 & 1 & -2\\ 3 & 1 & 5\end{bmatrix}$ and $B=\begin{bmatrix}1 & -1 & 5\\ -2 & 0 & 6\\ 3 & 2 & 4\end{bmatrix}$ are given. Find the products.

a. AB b. BA

Solution a. The product AB is defined because A has dimension 2×3 and B has dimension 3×3. Moreover, the product AB will have dimension 2×3. Look at the procedure for calculating the product:

$2 \cdot 1 + 1 \cdot (-2) + (-2) \cdot 3 = -6$

1st row, 1st column $\begin{bmatrix} 2 & 1 & -2 \\ 3 & 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 5 \\ -2 & 0 & 6 \\ 3 & 2 & 4 \end{bmatrix} = \begin{bmatrix} -6 & & \\ & & \end{bmatrix}$

$2 \cdot (-1) + 1 \cdot 0 + (-2) \cdot 2 = -6$

1st row, 2nd column $\begin{bmatrix} 2 & 1 & -2 \\ 3 & 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 5 \\ -2 & 0 & 6 \\ 3 & 2 & 4 \end{bmatrix} = \begin{bmatrix} -6 & -6 & \\ & & \end{bmatrix}$

$2 \cdot 5 + 1 \cdot 6 + (-2) \cdot 4 = 8$

1st row, 3rd column $\begin{bmatrix} 2 & 1 & -2 \\ 3 & 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 5 \\ -2 & 0 & 6 \\ 3 & 2 & 4 \end{bmatrix} = \begin{bmatrix} -6 & -6 & 8 \\ & & \end{bmatrix}$

2nd row, 1st column $\begin{bmatrix} 2 & 1 & -2 \\ 3 & 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 5 \\ -2 & 0 & 6 \\ 3 & 2 & 4 \end{bmatrix} = \begin{bmatrix} -6 & -6 & 8 \\ 16 & & \end{bmatrix}$

$3 \cdot 1 + 1 \cdot (-2) + 5 \cdot 3 = 16$

2nd row, 2nd column $\begin{bmatrix} 2 & 1 & -2 \\ 3 & 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 5 \\ -2 & 0 & 6 \\ 3 & 2 & 4 \end{bmatrix} = \begin{bmatrix} -6 & -6 & 8 \\ 16 & 7 & \end{bmatrix}$

$3 \cdot (-1) + 1 \cdot 0 + 5 \cdot 2 = 7$

2nd row, 3rd column $\begin{bmatrix} 2 & 1 & -2 \\ 3 & 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 5 \\ -2 & 0 & 6 \\ 3 & 2 & 4 \end{bmatrix} = \begin{bmatrix} -6 & -6 & 8 \\ 16 & 7 & 41 \end{bmatrix}$

$3 \cdot 5 + 1 \cdot 6 + 5 \cdot 4 = 41$

So the product is $\begin{bmatrix} 2 & 1 & -2 \\ 3 & 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 5 \\ -2 & 0 & 6 \\ 3 & 2 & 4 \end{bmatrix} = \begin{bmatrix} -6 & -6 & 8 \\ 16 & 7 & 41 \end{bmatrix}$.

b. Since the dimensions of B and A are 2×2 and 3×2 respectively, the product BA is not defined. This shows us that matrix multiplication is not always commutative: $AB \neq BA$.

Check Yourself

Find the products.

a. $\begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix}\begin{bmatrix} -2 & 4 & 2 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix}$ b. $\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ c. $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$

Answers

a. $\begin{bmatrix} -5 & 7 & -1 \\ -3 & 6 & 6 \end{bmatrix}$ b. $\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$ c. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

EXAMPLE 11 Given the matrices $A=\begin{bmatrix} 0 & 3 \\ -2 & 6 \end{bmatrix}$ and $B=\begin{bmatrix} -1 & 7 \\ 2 & 4 \end{bmatrix}$, show that $AB \neq BA$.

Solution Using the definition of the product of two matrices, we get

$$AB=\begin{bmatrix} 0 & 3 \\ -2 & 6 \end{bmatrix}\begin{bmatrix} -1 & 7 \\ 2 & 4 \end{bmatrix}=\begin{bmatrix} 0\cdot(-1)+3\cdot 2 & 0\cdot 7+3\cdot 4 \\ (-2)\cdot(-1)+6\cdot 2 & (-2)\cdot 7+6\cdot 4 \end{bmatrix}=\begin{bmatrix} 6 & 12 \\ 14 & 10 \end{bmatrix}$$

$$BA=\begin{bmatrix} -1 & 7 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ -2 & 6 \end{bmatrix}=\begin{bmatrix} (-1)\cdot 0+7\cdot(-2) & (-1)\cdot 3+7\cdot 6 \\ 2\cdot 0+4\cdot(-2) & 2\cdot 3+4\cdot 6 \end{bmatrix}=\begin{bmatrix} -14 & 39 \\ -8 & 30 \end{bmatrix},$$

Matrix multiplication is not in general commutative: $AB \neq BA$.

Thus $AB \neq BA$.

EXAMPLE 12 The following table shows the probabilities of a taxi ride ending at each of three destinations for taxis traveling among three sections of a city. For example, the probability of picking up a rider southside and dropping him off downtown is 30%.

Pickup	Destination		
	Northside	Downtown	Southside
Northside	50%	20%	30%
Downtown	10%	40%	50%
Southside	30%	30%	40%

What is the probability of starting downtown and being downtown again after two taxi rides?

Solution Let us represent the information we are given in a matrix

$$\begin{array}{c} \\ N \\ D \\ S \end{array}\begin{array}{c} \begin{array}{ccc} N & D & S \end{array} \\ \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.1 & 0.4 & 0.5 \\ 0.3 & 0.3 & 0.4 \end{bmatrix} \end{array} = P.$$

Then $P^2 = P \cdot P = \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.1 & 0.4 & 0.5 \\ 0.3 & 0.3 & 0.4 \end{bmatrix}\begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.1 & 0.4 & 0.5 \\ 0.3 & 0.3 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.36 & 0.27 & 0.37 \\ 0.24 & 0.33 & 0.43 \\ 0.3 & 0.3 & 0.4 \end{bmatrix}$

The second column of the product matrix P^2 shows the probability of ending up downtown after two trips, given the original starting point of either N, D or S. The Downtown to Down town probability is in the middle: 0.33, or 33%.

EXAMPLE 13 Given $X = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix}$, find XY, YX, X^2 and Y^2.

Solution

$$XY = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 2\cdot 0+0\cdot 2 & 2\cdot 1+0\cdot(-3) \\ 1\cdot 0+(-1)\cdot 2 & 1\cdot 1+(-1)\cdot(-3) \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 4 \end{bmatrix}$$

$$YX = \begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0\cdot 2+1\cdot 1 & 0\cdot 0+1\cdot(-1) \\ 2\cdot 2+(-3)\cdot 1 & 2\cdot 0+(-3)\cdot(-1) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$$

$$X^2 = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2\cdot 2+0\cdot 1 & 2\cdot 0+0\cdot(-1) \\ 1\cdot 2+(-1)\cdot 1 & 1\cdot 0+(-1)\cdot(-1) \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 1 & 1 \end{bmatrix}$$

$$Y^2 = \begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 0\cdot 0+1\cdot 2 & 0\cdot 1+1\cdot(-3) \\ 2\cdot 0+(-3)\cdot 2 & 2\cdot 1+(-3)\cdot(-3) \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -6 & 11 \end{bmatrix}$$

EXAMPLE 14 Solve $\begin{bmatrix} -4 & x \\ -x & 4 \end{bmatrix}^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

Solution First expand the left-hand side:

$$\begin{bmatrix} -4 & x \\ -x & 4 \end{bmatrix}^2 = \begin{bmatrix} -4 & x \\ -x & 4 \end{bmatrix}\begin{bmatrix} -4 & x \\ -x & 4 \end{bmatrix} = \begin{bmatrix} 16-x^2 & 0 \\ 0 & 16-x^2 \end{bmatrix}$$

Now equate corresponding entries:

$$16 - x^2 = -1$$
$$x^2 = 17$$

$x = \pm\sqrt{17}$. This is the solution.

Check Yourself

1. Calculate the products.

   a. $\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & -6 \\ 5 & 7 \end{bmatrix}$ b. $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

2. Given $\begin{bmatrix} 2 & 0 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$, find A^2 and A^3.

3. Given the function $f(x) = x^2 - 5x + 2I$ where I is the identity matrix, calculate $f(A)$ for $A = \begin{bmatrix} 2 & 0 \\ 4 & 5 \end{bmatrix}$.

Answers

1. a. $\begin{bmatrix} 14 & -11 \\ 7 & 20 \end{bmatrix}$ b. $\begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix}$ 2. $A^2 = \begin{bmatrix} 4 & 6 & 6 \\ 2 & 2 & 3 \\ 2 & 0 & 2 \end{bmatrix}$, $A^3 = \begin{bmatrix} 14 & 12 & 18 \\ 6 & 6 & 8 \\ 4 & 4 & 6 \end{bmatrix}$ 3. $\begin{bmatrix} -4 & 0 \\ 8 & 2 \end{bmatrix}$

E- DETERMINANT

This formula is one you should memorize: To obtain the determinant of a 2 by 2 matrix, subtract the product of the offdiagonal entries from the product of the diagonal entries:

$$\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

To illustrate,

$$\det\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = (1)(4) - (2)(3) = -2$$

Using the notation for these permutations given in Example 1, as well as the evaluation of their signs in Example 3, the sum above becomes

$$\begin{aligned} \det A = {} & (1+)a_{11}a_{12}a_{33} + (-1)a_{11}a_{23}a_{23} \\ & +(-1)a_{11}a_{21}a_{33} + (+1)a_{12}a_{23}a_{31} \\ & +(+1)a_{13}a_{21}a_{32} + (+1)a_{13}a_{22}a_{31} \end{aligned}$$

or, more simply,

$$\det A = a_{11}a_{12}a_{33} + a_{11}a_{22}a_{33}$$
$$+a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} \quad (**)$$

As you can see, there is quite a bit of work involved in computing a determinant of an n by n matrix directly from definition (*), particularly for large n. In applying the definition to evaluate the determinant of a 7 by 7 matrix, for example, the sum (*) would contain more than five thousand terms. This is why no one ever actually evaluates a determinant by this laborious method.

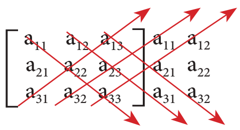
A simple way to produce the expansion (**) for the determinant of a 3 by 3 matrix is first to copy the first and second columns and place them after the matrix as follows:

$$A = \begin{bmatrix} a_{11}a_{12}a_{13} \\ a_{21}a_{22}a_{23} \\ a_{31}a_{32}a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

Then, multiply down along the three diagonals that start with the first row of the original matrix, and multiply up along the three diagonals tha start with the bottom row of the original matrix. Keep the signs of the three "down" products, reverse the signs of the three "up" products, and add all six resulting terms; this gives (**) Note: This method works only for 3 by 3 matrices.

reverse the signs of these products



keep the signs of these products

F- GRAMER`S RULE

Cramer's Rule
Consider the general 2 by 2 linear system

$$a_{11}\chi\ a_{12}y = b_1$$
$$a_{21}\chi\ a_{22}\ y = b_1$$

Multiplying the first equation by a 22, the second by – a 12, and adding the results eliminates y and permits evaluation of x:

$$a_{11}a_{22}\chi + a_{12}a_{22}\,y = a_{22}b_1$$
$$-a_{12}a_{21}\chi - a_{12}a_{22}\,y = -\ a_{12}\,b_2$$

$$\chi\,(a_{12}a_{22} - a_{12}a_{21}) = a_{22}b_1 - a_{21}\,b_2$$

$$\chi = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}$$

assuming that a 11 a 22 – a 12 a 21 ≠ 0. Similarly, multiplying the first equation by – a 21, the second by a 11, and adding the results eliminates x and determines y:

$$-a_{11}a_{21}\chi - a_{12}a_{21}\,y = -\ a_{21}b_1$$
$$a_{11}a_{21}\,\chi + a_{11}a_{22}\,y = a_{11}\,b_2$$

$$y\,(a_{11}a_{22} - a_{12}a_{21}) = a_{11}b_2 - a_{21}\,b_1$$

$$y = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}$$

again assuming that a 11 a 22 – a 12 a 21 ≠ 0. These expressions for x and y can be written in terms of determinants as follows:

$$\chi = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}} = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{22} \\ a_{12} & a_{21} \end{vmatrix}}$$

and

$$y = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}} = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

If the original system is written in matrix form,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \chi \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

then the denominators in the above expressions for the unknowns x and y are both equal to the determinant of the coefficient matrix. Furthermore, the numerator in the expression for the first unknown, x, is equal to the determinant of the matrix that results when the first column of the coefficient matrix is replaced by the column of constants, and the numerator in the expression for the second unknown, y, is equal to the determinant of the matrix that results when the second column of the coefficient matrix is replaced by the column of constants. This is Cramer's Rule for a 2 by 2 linear system.
Extending the pattern to a 3 by 3 linear system,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \chi \\ y \\ z \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Cramer's Rule says that if the determinant of the coefficient matrix is nonzero, then expressions for the unknowns x, y, and z take on the following form:

$$\chi = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{22} & a_{13} \\ a_{12} & a_{21} & a_{23} \\ a_{33} & a_{32} & a_{33} \end{vmatrix}} \quad y = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{22} & a_{13} \\ a_{12} & a_{21} & a_{23} \\ a_{33} & a_{32} & a_{33} \end{vmatrix}} \quad Z = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{22} & a_{13} \\ a_{12} & a_{21} & a_{23} \\ a_{33} & a_{32} & a_{33} \end{vmatrix}}$$

The general form of Cramer's Rule reads as follows: A system of n linear equations in n unknowns, written in matrix form A x = b as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1a} \\ a_{21} & a_{22} & \cdots & a_{2a} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{m} \end{bmatrix} \begin{bmatrix} \chi_1 \\ \chi_2 \\ \vdots \\ \chi_{n1} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

will have a unique solution if det A ≠ 0, and in this case, the value of the unknown x j is given by the expression

$$\chi_j = \frac{\det A_j}{\det A}$$

Example1:

Finde the solution of the linear equation:

$2x - 3y = -4 \qquad , 3x + y = 2$

solution:

$$x = \frac{detx}{det} = \frac{\begin{vmatrix} -4 & -3 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & -3 \\ 3 & 1 \end{vmatrix}} = \frac{-4+6}{2+9} = \frac{2}{11}$$

$$y = \frac{dety}{det} = \frac{\begin{vmatrix} 2 & -4 \\ 3 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & -3 \\ 3 & 1 \end{vmatrix}} = \frac{4+12}{2+9} = \frac{16}{11}$$

Example2:

Finde the solution of the linear equation:

x+ 3y -z =1

2x +2y +z = 0

3x +y +2z = -1

solution: we finde the det of variables

$$\det = \begin{vmatrix} 1 & 3 & -1 \\ 2 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix} = 4$$

$$\det x = \begin{vmatrix} 1 & 3 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & 2 \end{vmatrix} = -2$$

$$\det y = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 3 & -1 & 2 \end{vmatrix} = 2$$

$$\det z = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 2 & 0 \\ 3 & 1 & -1 \end{vmatrix} = 0$$

$$x = \frac{\det x}{\det} = \frac{-2}{4} = \frac{-1}{2}$$

$$y = \frac{\det y}{\det} = \frac{2}{4} = \frac{1}{2}$$

$$z = \frac{\det z}{\det} = \frac{0}{4} = 0$$

EXERCISES

D. Operations on Matrices

1. Calculate $A+B$, $A-B$, $2A$ and $2A-B$ for each pair of matrices.

   a. $A=\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}$, $B=\begin{bmatrix}-3 & -2\\ 4 & 2\end{bmatrix}$

   b. $A=\begin{bmatrix}6 & -1\\ 2 & 4\\ -3 & 5\end{bmatrix}$, $B=\begin{bmatrix}1 & 4\\ -1 & 5\\ 1 & 10\end{bmatrix}$

   c. $A=\begin{bmatrix}2\\ -1\end{bmatrix}$, $B=\begin{bmatrix}6\\ 2\end{bmatrix}$

2. Find c_{21} and c_{13} if $C=3A-2B$,

   $A=\begin{bmatrix}5 & 4 & 4\\ -3 & 1 & 2\end{bmatrix}$ and $B=\begin{bmatrix}1 & 2 & -7\\ 0 & -5 & 1\end{bmatrix}$.

3. Find c_{23} and c_{32} if $C=2A+5B$,

   $A=\begin{bmatrix}4 & 11 & -9\\ 0 & 3 & 2\\ -3 & 1 & 1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 2 & -7\\ -4 & 6 & 11\\ -6 & 4 & 9\end{bmatrix}$.

4. Solve the matrix equation for a, b and c.

   $$4\begin{bmatrix}a & b\\ c & -1\end{bmatrix}=2\begin{bmatrix}b & c\\ -a & 1\end{bmatrix}+2\begin{bmatrix}4 & a\\ 5 & -a\end{bmatrix}$$

5. Find k, m and n if

   $$\begin{bmatrix}n & k\\ 1 & -3\end{bmatrix}=\begin{bmatrix}-4 & 3\\ 2 & -1\end{bmatrix}+2\begin{bmatrix}1 & n\\ m & k\end{bmatrix}.$$

6. Solve the matrix equations for a, b, c and d.

   a. $\begin{bmatrix}a-b & 2b+c\\ c-2b & a+d\end{bmatrix}=\begin{bmatrix}5 & 1\\ 5 & 5\end{bmatrix}$

   b. $\begin{bmatrix}a+b & b+c\\ a-c & b-a\end{bmatrix}=\begin{bmatrix}0 & 3\\ -3 & 2\end{bmatrix}$

7. Calculate AB in each case.

   a. $A=\begin{bmatrix}2 & -1\\ 1 & 4\end{bmatrix}$, $B=\begin{bmatrix}0 & 0\\ 3 & -3\end{bmatrix}$

   b. $A=\begin{bmatrix}3 & -1\\ 1 & 3\end{bmatrix}$, $B=\begin{bmatrix}1 & -3\\ 3 & 1\end{bmatrix}$

   c. $A=\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$, $B=\begin{bmatrix}1 & 3\\ -3 & 1\end{bmatrix}$

   d. $A=\begin{bmatrix}1 & -1 & 7\\ 2 & -1 & 8\\ 3 & 1 & -1\end{bmatrix}$, $B=\begin{bmatrix}1 & 1 & 2\\ 2 & 1 & 1\\ 1 & -3 & 2\end{bmatrix}$

   e. $A=[3 \;\; 2 \;\; 1]$, $B=\begin{bmatrix}2\\ 3\\ 0\end{bmatrix}$

8. Calculate AB in each case.

a. $A=\begin{bmatrix} 2 & 1 \\ -2 & 4 \\ 1 & 6 \end{bmatrix}$, $B=\begin{bmatrix} 0 & -1 & 0 \\ 4 & 0 & 2 \\ 8 & -1 & 7 \end{bmatrix}$

b. $A=\begin{bmatrix} 0 & -1 & 0 \\ 4 & 0 & 2 \\ 8 & -1 & 7 \end{bmatrix}$, $B=\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$

c. $A=\begin{bmatrix} -1 & 3 \\ 4 & -5 \\ 0 & 2 \end{bmatrix}$, $B=\begin{bmatrix} 1 & 2 \\ 0 & 7 \end{bmatrix}$

d $A=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$, $B=\begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

e. $A=\begin{bmatrix} 0 & 0 & -5 \\ 0 & 0 & -3 \\ 0 & 0 & 7 \end{bmatrix}$, $B=\begin{bmatrix} 6 & -11 & 4 \\ 8 & 16 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ f. $A=\begin{bmatrix} 6 \\ -2 \\ 1 \\ 6 \end{bmatrix}$, $B=[10$

E - Determinant:

9.Find the det.

i) $\begin{vmatrix} 5 & 4 \\ 0 & 6 \end{vmatrix}$ ii) $\begin{vmatrix} -7 & 13 \\ 13 & -7 \end{vmatrix}$ iii) $\begin{vmatrix} 2 & -1 & 6 \\ 1 & 0 & 1 \\ 5 & 0 & 1 \end{vmatrix}$ iv) $\begin{vmatrix} 3 & 0 & 6 \\ 4 & 0 & 7 \\ 5 & 0 & 8 \end{vmatrix}$

F - Gramer`s Rule

10. Finde the solution of the linear equation: by using Gramer`s Rule

i) $2x = 3y + 4$
$5y = -4x - 1$

ii) $6L - 7K = 0$
$4L + 3K = 0$

iii) $-x + 3y + z = 0$
$3x - 2y - z = 1$
$x + y + 2z = 0$

تمّ بحمده